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Some New Classes of General Harmonic-like Variational Inequalities

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Abstract

Several new classes of general harmonic-like variational inequalities involving two arbitrary operators are introduced and considered in this paper. Some important cases are discussed, which can be obtained by choosing suitable and appropriate choice of the operators. Projection technique is applied to establish the equivalence between the general harmonic-like variational inequalities and fixed point problems. This alternative formulation is used to discuss the uniqueness of the solution as well as to propose a wide class of proximal point algorithms. Convergence criteria of the proposed methods is considered. Asymptotic stability of the solution is studied using the first order dynamical system associated with variational inequalities. Second order dynamical systems associated with general quasi variational inequalities are applied to suggest some inertial type methods. Some special cases are discussed as applications of the main results. We also show that the change of variable can be used to show that the harmonic-like variational inequalities. Several open problems are indicated for future research work.

1 Introduction

Stampacchia [61] proved that the minimum of a differentiable convex function associated with obstacle problem in potential theory can be characterized by an inequality. Motivated and inspired by these facts, Lions and Stampacchia [23] considered and studied the variational inequalities. Variational inequality theory can be viewed as a novel extension and generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. It is amazing that a wide class of unrelated problems can be studied in unified framework of variational inequalities, which occur in various branches of pure and applied sciences. For more details, see [1, 2, 8, 11, 14–20, 22–25, 29–49, 51–56, 58, 59, 61–64].

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It is worth mentioning that variational inequalities theory is clasely related to the convexity theory contains a wealth of novel ideas and techniques. These theories have played the significant role in the development of almost all the branches of pure and applied sciences. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions and convex sets. Anderson et al. [2] have investigated several aspects of the harmonic convex functions. Harmonic analysis has become a vast subject with applications in areas as diverse as number theory, representation theory, signal processing, quantum mechanics, tidal analysis, spectral analysis, and neuroscience. The harmonic mean is generally used when there is a necessity to give greater weight to the smaller items. The harmonic mean is often used to calculate the average of the ratios or rates of the given values. It is the most appropriate measure for ratios and rates because it equalizes the weights of each data point. The harmonic mean plays a crucial role in numerous scientific and engineering fields, particularly in semiconductor physics. In this context, the effective mass of charge carriers within a semiconductor is often computed using the harmonic mean of its crystallographic directions. Specifically, the conductivity effective mass in a semiconductor is determined by the harmonic mean of its three principal crystallographic axes. This property is vital for understanding and optimizing electronic materials' conductivity and performance. Beyond electrical circuits and semiconductor physics, harmonic convexity also has implications in signal processing. Functions that exhibit harmonic convexity are associated with higher frequency components that can distort fundamental waveforms, leading to undesired oscillations. This property is particularly relevant in applications involving waveform stability and signal transmission, where minimizing undesirable frequency components is essential. From a mathematical perspective, harmonic convex functions have been instrumental in optimization and variational inequalities. Noor et al. [41, 43, 44] have shown that the minimum of the differentiable harmonic convex functions on the harmonic convex set can be characterized by a class of harmonic variational inequalities. Their work has paved the way for advancements in solving optimization problems, where harmonic convexity plays a central role. Recent research has further expanded the scope of harmonic convexity, see the refrences [2, 43–47, 47–50, 53].

The harmonic means have novel applications in electrical circuits theory. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if η_1 and η_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula $(\frac{1}{\eta_1}+\frac{1}{\eta_2})^{-1}=\frac{\eta_1\eta_2}{\eta_1+\eta_2}$, which is half the harmonic means. The harmonic mean is employed in finance to determine the average multiples like the price-income ratio, etc. It is also practised by market professionals in order to determine the patterns like Fibonacci Sequences. Harmonic means are also used to average data like price multiples." The Asian options with harmonic average have been considered in Al-Azemi et al. [6], which can be viewed as a new direction in the study of the risk analysis stock exchange and financial mathematics. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al. [47, 50] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions.

One of the most difficult and important problems in quasi variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inclusions and their variant forms. These methods have been extended and modified in numerous ways for solving the variational inclusions and their variant forms. Noor [36, 37] suggested and analyzed several three-step forward-backward splitting algorithms for solving variational inequalities by using the updating technique. These three-step methods are also known as Noor's iterations. It is noted that these forward-backward splitting algorithms are similar to those of Glowinski et al. [15], which they suggested by using the Lagrangian technique. Glowinski et al. [16] discussed the convergence analysis and applications of the Glowinski-Le Tallec splitting method. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. For applications of the splitting techniques to partial differential equations. In recent years, considerable interest has been shown in developing various extensions and generalizations of Noor iterations, both for their own sake and for their applications. In passing, we point out that the three-step iterative methods are also known as Noor iterations, which contain Picard method, Mann (one step) iteration, Ishikawa (two-step) iterations as special cases. It have shown the Noor orbit demonstrates that the boundary of the fixed point region is similar to natural features such as bird nests and certain types of peacock wing structures. This is demonstrated by geometrical and numerical analysis of composite Julia sets and composite Mandelbrot sets for the Noor iteration, see et al. [27] and Negi [28]. Recently, Noor (three step) iterations have been generalized and extended in various directions using innovative ideas to explore their applications in fractal, chaos, images, signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing, solar energy optimizations and image in painting. For novel applications, modifications and generalizations of the Noor iterations, see [7, 9, 10, 22, 26–28, 56, 58, 59, 62, 64] and the references therein.

The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [14]. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. This equivalent formulation is useful in studying the asymptotic stability of the solution of the variational inequality applying the Lyapunov theory of functional differential equations. This dynamical system is a first order initial value problem. Discretizing the dynamical system and using the finite difference idea, Noor et. al. [42] have been shown that the dynamical system can be used to suggest some implicit iterative method for solving various classes of variational inequalities. For the applications and numerical methods applying the dynamical systems, see [19, 25, 40, 42, 52, 53, 63] and the references therein.

It is well known that the projection, descent method and their variant forms cannot be applied to consider the existence of the solution and to suggest some iterative type methods for some kind of variational inequalities due to their complicated and complex nature. To overcome these drawbacks, which is mainly due to Lions et al. [23]. Glowinski et al [15] used this technique to discuss the existence of the mixed variational inequalities. In this technique, one consider an auxiliary problem related to the origin problem. This way, one defines a mapping connection these problem and proves that this mapping has a fixed point satisfying the original problem. Noor [30, 33, 37] and Noor et al. [43–47, 52–54] have used this technique for solving mixed variational inequalities and equilibrium problems. One can easily show that the projection methods and its variant forms can be obtained as special cases of the auxiliary principle.

Motivated and inspired by the harmonic-like going research in these dynamic and active areas, we consider some new classes of harmonic-like variational inequalities involving two arbitrary operators. For appropriate and suitable choice of the operators, convex set and the space, we can obtain the inverse harmonic-like variational inequalities, quasi complementarity problems and variational inequalities as special cases. Making use of the best approximation result, we show that the general harmonic-like variational inequalities are equivalent to the fixed point problems. We use this alternative formulation to discuss the unique existence of the solution. Several multi step proximal point methods are proposed and investigated for solving the general quasi variational inequalities applying the fixed point, Wiener-Hopf, auxiliary principle and dynamical system. These methods include the Mann (one-step) iteration, Ishikawa (two-step) iteration, Noor (three-step)iteration and forward-backward splitting methods for finding the approximate solution. Convergence criteria is investigated under suitable conditions. We also considered the second order boundary value problem related to the variational inequalities coupled with dynamical system. Using the finite difference forward and backward interpolation, proximal point methods are proposed. It is shown that the change of variable methods can be used to establish the equivalence between the harmonic-like variational inequalities and fixed point method, which is another novel technique. We have only investigated the theoretical aspect of the iterative methods. Developments of the numerical applicable methods need further research efforts and can be considered an important open problems. Since the general harmonic-like variational inequalities include the general variational inequalities, variational inequalities and complementarity problems as special cases, our result continue to hold for these problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Formulations and Basic Facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively.

First of all, we now show that the minimum of a differentiable harmonic-like convex function on a harmonic-like convex set Ω in $\mathcal{H}$ can be characterized by the harmonic-like variational inequalities. For this purpose, we recall the following well known concepts and results [12, 29, 47, 50].

Definition 2.1. *A set Ω is said to be a harmonic-like convex set, if*

$$(\frac{2w\mu}{w+\mu}) + t(\nu-\mu) \in \Omega, \quad \forall \mu, w, \nu \in \Omega, \quad \mu \leq w \leq \nu, \quad \mu \leq w \leq \nu, \quad t \in [0,1].$$

Definition 2.2. *A function $F : \Omega \longrightarrow \mathcal{H}$ is said to be a harmonic-like convex, if*

$$F((\frac{2w\mu}{w+\mu}) + t(\nu-\mu)) \leq (1-t)F(\mu) + tF(\nu), \quad \forall w, \mu, \nu \in \Omega, \quad \mu \leq w \leq \nu, \quad t \in [0,1].$$

Remark 2.1. *Here $w \in [\mu, \nu]$ can be viewed as the weight function.*
(a). If $w = \mu$, then the definitions 2.1 and 2.2 are exactly the convex set and convex functions.
(b). For $w = \nu$, the harmonic-like convex set and harmonic-like functions reduce to harmonic mean set and harmonic mean function respectively.

Lemma 2.1. *Let $F : \Omega \longrightarrow \mathcal{H}$ be a differentiable harmonic-like convex function. Then $\mu \in \Omega$ is the minimum of harmonic-like convex function F on Ω, if and only if, $\mu \in \Omega$ satisfies the inequality*

$$\langle F'(\frac{2w\mu}{w+\mu}), \nu-\mu\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{2.1}$$

where $F'(\frac{2w\mu}{w+\mu})$ is the differential of F.

Proof. Let $\mu \in \Omega$ be a minimum of a differentiable harmonic-like convex function F on Ω. Then

$$F(\mu) \leq F(\nu), \quad \forall v \in \Omega. \tag{2.2}$$

Since Ω is a harmonic-like convex set, so, for all $\mu, w, \nu \in \Omega, t \in [0,1], v_t = (\frac{2w\mu}{w+\mu}) + t(\nu-\mu) \in \Omega$.

Setting $\nu = v_t$ in (2.2), we have

$$F(\mu) \leq F((\frac{2w\mu}{w+\mu}) + t(\nu-\mu)).$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'((\frac{2w\mu}{w+\mu}), \nu-\mu\rangle \geq 0, \quad \forall w, \nu \in \Omega,$$

which is the required result(2.1).

Conversely, let $\mu \in \Omega$ satisfy the inequality (2.1). Since F is a harmonic-like convex function, *so* $\forall \mu, \nu \in \Omega \quad t \in [0,1], \quad (\frac{2w\mu}{w+\mu}) + t(\nu-\mu) \in \Omega$ and

$$F((\frac{2w\mu}{w+\mu}) + t(\nu-\mu)) \leq (1-t)F(\mu) + tF(\nu),$$

which implies that

$$F(\nu) - F(\mu) \geq \frac{F((\frac{2w\mu}{w+\mu}) + t(\nu - \mu)) - F(\mu)}{t}.$$

Letting $t \longrightarrow 0$, and using (2.1), we have

$$F(\nu - F(\mu) \geq \langle F'(\frac{2w\mu}{w+\mu}), \nu - \mu\rangle \geq 0,$$

which implies that

$$F(\mu) \leq F(\nu), \quad \forall \in \Omega$$

showing that $\mu \in \Omega$ is the minimum of F on Ω in $\mathcal{H}$. □

The inequality of the type (2.1) is called the harmonic-like variational inequality. It is known that the problem (2.1) may not arise as the optimality conditions of the differentiable convex functions. This motivated us to consider a more general problem of which the problem (2.1) is a special case. To be more precise, for given operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$, consider the problem of finding $\mu \in \Omega \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), \nu - \mu\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.3}$$

which is called the harmonic-like variational inequality.

We now introduce the problem of general variational inequality. Let $\Omega \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ be a closed convex set. Then, for given operators $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) + \mu - g(\mu), \nu - \mu\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.4}$$

which is called the general harmonic-like variational inequality.

Special Cases

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2.4).

(I). If $g(\mu) = \mu$, then problem (2.4) reduces to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}(\frac{2wg(\mu)}{w+g(\mu)}, \nu - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.5}$$

is called the general harmonic-like variational inequality.

(II). If $\mathcal{T} = I$, the identity operator, then problem (2.5) reduces to finding $\mu \in \Omega$ such that

$$\langle (\frac{2w\mu}{w+\mu}), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.6}$$

This inequality is called the inverse harmonic-like variational inequality.

(III). If $g = I, \quad \mu = w$, then the problem (2.4) collapses to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.7}$$

which is called variational inequality, introduced by Lions and Stamapcchia [23] in the impulse control theory. For the numerical analysis, sensitivity analysis, dynamical systems and other aspects of variational inequalities and related optimization programming problems. See [1, 2, 8, 11, 14–20, 22–25, 29–49, 51–56, 58, 59, 61–64] and the references therein.

(IV). For the polar cone $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu \rangle \geq 0, \quad \forall \nu \in \Omega\}$, the problem (2.3) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\mu \in \Omega, \quad \mathcal{T}(\frac{2w\mu}{w+\mu}) \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}(\frac{2w\mu}{w+\mu}), g(\mu) \rangle = 0, \tag{2.8}$$

is called the general harmonic-like complementarity problem. Obviously general harmonic-like complementarity problems include the general complementarity problems, nonlinear complementary problems and linear complementarity problems, which were introduced and studied in Al-said et al. [1], Cottle et al. [11], Noor [31, 33, 37] and Noor et al. [52, 53, 55] in game theory, management sciences and quadratic programming as special cases. This inter relations among these problems have played a major role in developing numerical results for these problems and their applications.

Remark 2.2. *It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, g$, closed convex set Ω and the spaces, one can obtain several classes of variational inequalities, complementarity problems and optimization problems as special cases of the nonlinear harmonic-like variational inequalities (2.4). This shows that the problem (2.4) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the general quasi-variational inequalities and their variants.*

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the general quasi variational inequalities and the fixed point problems. This result is used in the analysing the convergence analysis of the projective implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.2. *[20] Let Ω be a closed and convex set in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.9}$$

if and only if,

$$\mu = \Pi_\Omega(z),$$

where Π_Ω *is projection of* $\mathcal{H}$ *onto the closed convex-valued set* Ω.

It is well known that the projection operator Π_Ω is nonexpansive, that is,

Assumption 2.1.

$$\|\Pi_\Omega\mu - \Pi_\Omega\nu\| \leq \|\mu - \nu\|, \forall \mu, \nu \in \mathcal{H}. \tag{2.10}$$

Assumption 2.1 has been used to prove the existence of a solution of general variational inequalities as well as in analyzing convergence of the iterative methods.

Definition 2.3. *[47, 50] An operator* $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ *is said to be:*

1. *Strongly harmonic-like monotone, if there exist a constant* $\alpha > 0$*, such that*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), \mu - \nu\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

2. *Harmonic-like Lipschitz continuous, if there exist a constant* $\beta > 0$*, such that*

$$\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\nu\| \leq \beta\|\mu - \nu\|, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

3. *Monotone, if*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\nu, \mu - \nu\rangle \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

4. *Pseudo monotone, if*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), \nu - \mu\rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}(\frac{2w\nu}{w+\nu}), \nu - \mu\rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.3. *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

For $w = \mu$, Definition 2.3 reduces to the strongly mononivity, Lipschitz continuity and pseudo-monotonicity of the operator.

3 Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the general harmonic-like variational inequalities.

Using Lemma 2.2, one can show that the general harmonic-like variational inequalities are equivalent to the fixed point problems.

Lemma 3.1. [22] *The function $\mu \in \Omega$ is a solution of the general harmonic-like variational inequality (2.4), if and only if, $\mu \in \Omega$ satisfies the relation*

$$\mu = \Pi_{\Omega(\mu)}[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})], \tag{3.1}$$

where Π_{Ω} is the projection operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \Omega$ be the problem (2.4). Then

$$\langle \rho T(\frac{2w\mu}{w+\mu}) + \mu - g(\mu), \nu - \mu\rangle \geq 0, \quad \forall w, \nu \in \Omega.$$

Using Lemma 2.2, we have

$$\mu = \Pi_{\Omega(\mu)}[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})],$$

the required result. □

Lemma 3.1 implies that the general harmonic-like variational inequality (2.4) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation (3.1) will play an important role in deriving the main results.

From the equation (3.1), we have

$$\mu = \Pi_{\Omega}[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})].$$

We define the function Φ associated with (3.1) as

$$\Phi(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})]. \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.4), it is enough to show that the map Φ defined by (3.2) has a fixed point.

Theorem 3.1. *Let the operator $\mathcal{T}$ harmonic-like Lipschitz continuous with constants $\beta > 0$ and the operator g be Lipschitz continuous with constant $\zeta > 0$, respectively. If Assumption 2.1 holds and there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-\zeta}{\beta}, \quad \zeta < 1, \tag{3.3}$$

where

$$\theta = \frac{1-\zeta}{\beta},$$

then there exists a unique solution of the problem (2.4).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.4) are equivalent. Thus it is enough to show that the map $\Phi(\mu)$, defined by (3.2) has a fixed point.

For all $\nu \neq \mu \in \Omega$, we have

$$\begin{aligned} \|\Phi(\mu) - \Phi(\nu)\| &= \Pi_\Omega\|[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})] - \Pi_\Omega[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\| \\ &\leq \|g(\nu) - g(\mu) - \rho(\mathcal{T}(\frac{2w\nu}{w+\nu}) - \mathcal{T}(\frac{2w\mu}{w+\mu})\| \\ &\leq \|g(\nu) - g(\mu)\| + \rho\|\mathcal{T}(\frac{2w\nu}{w+\nu}) - \mathcal{T}(\frac{2w\mu}{w+\mu})\|. \end{aligned} \tag{3.4}$$

Since the operators $g, \mathcal{T}$ are Lipschitz continuous with constant $\zeta > 0$ and harmonic-like Lipschitz continuous with constant $\beta > 0$, respectively, it follows that

$$\begin{aligned} ||\Phi(\mu) - \Phi(\nu)|| &\leq \{\zeta + \rho\beta\}\,\|\mu - \nu\| \\ &= \theta\|\mu - \nu\|, \end{aligned}$$

where

$$\theta = (\zeta + \rho\beta). \tag{3.5}$$

From (3.3), it follows that $\theta < 1$, which implies that the map $\Phi(u)$ defined by (3.2) has a fixed point, which is the unique solution of (2.4). □

The fixed point formulation (3.1) is applied to propose and suggest the iterative methods for solving the problem (2.4).

This alternative equivalent formulation (3.1) is used to suggest the following iterative methods for solving the problem (2.4) using the updating technique of the solution.

Algorithm 3.1. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$y_n = \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \tag{3.6}$$

$$w_n = \Pi_\Omega[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})] \tag{3.7}$$

$$\mu_{n+1} = \Pi_\Omega[(w_n) - \rho\mathcal{T}(\frac{2ww_n}{w+w_n})]. \tag{3.8}$$

Algorithm 3.1 is a three step forward-backward splitting algorithm for solving general harmonic-like variational inequality (2.4).

We now study the convergence analysis of Algorithm 3.1, which is the main motivation of our next result.

Theorem 3.2. *Let the operators $\mathcal{T}, g$ satisfy all the assumptions of Theorem 3.1. If the condition (3.3) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3.1 converges to the exact solution $\mu \in \Omega$ of the general quasi variational inequality (2.4) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu \in \Omega(\mu)$ of the general quasi variational inequalities (2.4). Let $\mu \in \Omega$ be the unique solution of (2.4). Then, using Lemma 3.1, we have

$$\mu = \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] \tag{3.9}$$

$$= \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] \tag{3.10}$$

$$= \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]. \tag{3.11}$$

From (3.8), (3.9) and Assumption (2.1), we have

$$\begin{aligned} \|\mu_{n+1} - \mu\| &= \Pi_\Omega[g(w_n) - \rho\mathcal{T}(\frac{2ww_n}{w+w_n})w_n] - \Pi_{(\Omega}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\}\| \\ &\leq \|g(w_n) - g(\mu) - \rho(\mathcal{T}(\frac{2ww_n}{w+w_n})w_n - \mathcal{T}(\frac{2w\mu}{w+\mu}))\| \\ &\leq \theta\|w_n - \mu\|, \end{aligned} \tag{3.12}$$

where θ is defined by (3.5).

In a similar way, from (3.6) and (3.10), we have

$$\begin{aligned} \|w_n - \mu\| &\leq \theta\|y_n - \mu - (g(y_n) - g(\mu))\| \\ &+ \|y_n - \mu - \rho(\mathcal{T}(\frac{2wy_n}{w+\mu}) - \mathcal{T}(\frac{2w\mu}{w+\mu}))\| \\ &\leq \theta\|y_n - \mu\|, \end{aligned} \tag{3.13}$$

where θ is defined by (3.5).

From (3.6) and (3.11), we obtain

$$\|y_n - \mu\| \leq \theta\|\mu_n - \mu\|. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\|w_n - \mu\| \leq \theta\|\mu_n - \mu\|. \tag{3.15}$$

Form the above we equations, have

$$\|\mu_{n+1} - \mu\| \leq \theta\|\mu_n - \mu\|.$$

From (3.3), it follows that $\theta < 1$, Consequently the sequence $\{u_n\}$ converges strongly to μ. From (3.14), and (3.15), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We now suggest and analyze the three step scheme for solving the general harmonic-like variational inequality (2.4). These three step schemes were introduced and investigated by Noor [36, 37], which are called the Noor iterations.For the applications of novel Noor iterations in signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing and image in painting, see [3–5, 7, 9, 10, 22, 26–28, 48, 56, 58, 59, 62, 64] and the references therein.

Algorithm 3.2. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned}
y_n &= (1-\gamma_n)\mu_n + \gamma_n\Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\
w_n &= (1-\beta_n)y_n + \beta_n\Pi_\Omega[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})] \\
\mu_{n+1} &= (1-\alpha_n)w_n + \alpha_n\Pi_{\Omega(w_n)}[g(w_n) - \rho\mathcal{T}(\frac{2ww_n}{w+w_n})w_n].
\end{aligned}$$

Convergence analysis of Algorithm 3.2 can be studied using the above technique.

For $\gamma_n = 0$, Algorithm 3.2 reduces to:

Algorithm 3.3. *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned}
w_n &= (1-\beta_n)\mu_n + \beta_n\Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Pi_\Omega[g(w_n) - \rho\mathcal{T}(\frac{2ww_n}{w+w_n}),
\end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (2.4).

Note that for $\gamma_n = 0$ *and* $\beta_n = 0$, Algorithm 3.2 is called the Mann iterative method, that is,

Algorithm 3.4. *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\mu_{n+1} = (1-\beta_n)\mu_n + \beta\Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})].$$

We suggest another perturbed iterative scheme for solving the general harmonic-like variational inequality (2.4).

Algorithm 3.5. *For a given μ_o, compute the approximate solution $\{\mu_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n\Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] + \gamma_n h_n \\ w_n &= (1-\beta_n)y_n + \beta_n\Pi_\Omega[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})] + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)w_n + \alpha_n\Pi_\Omega[g(w_n) - \rho\mathcal{T}(\frac{2ww_n}{w+w_n}) + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and Π_Ω is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq 1; \quad \forall n \geq 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving general harmonic-like variational inequality (2.4).

Algorithm 3.6. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_\Omega[g(\mu_{n+1}) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \quad n = 0, 1, 2, ...$$

which is known as the modified projection method and is equivalent to the iterative method.

Algorithm 3.7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \Pi_{\Omega(\omega}[g(\omega_n) - \rho\mathcal{T}(\frac{2w\omega_n}{w+\omega_n})], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.4).

We can rewrite the equation (3.1) as:

$$\mu = \Pi_\Omega[g(\frac{\mu+\mu}{2}) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]. \tag{3.16}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 3.8. *[30]. For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_\Omega[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w + \mu_{n+1}})]. \tag{3.17}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (2.4).

Algorithm 3.9. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w + \mu_n})] \\ \mu_{n+1} &= \Pi_\Omega[g(\frac{\omega_n + \mu_n}{2}) - \rho\mathcal{T}(\frac{2w\omega_n}{w + \omega_n})]. \end{aligned}$$

From equation (3.1), we have

$$\mu = \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w(\mu + \mu)}{2w + \mu + \mu})]. \tag{3.18}$$

This fixed point formulation (3.18) is used to suggest the implicit method for solving the problem (2.4) as

Algorithm 3.10. *For a given $\mu_0 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w(\mu_n + \mu_{n+1})}{2w + \mu_n + \mu_{n+1}})]. \tag{3.19}$$

We can use the predictor-corrector technique to rewrite Algorithm 3.10 as:

Algorithm 3.11. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w + \mu_n})\mu_n], \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w(\mu_n + \omega_n)}{2w + \mu_m + \omega_n})]. \end{aligned}$$

is known as the mid-point implicit method for solving the problem (2.4).

We again use the above fixed formulation to suggest the following implicit iterative method.

It is obvious that the above Algorithms have been suggested using different variant of the fixed point formulations (3.1). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the problem (2.4) and related optimization problems.

One can rewrite (3.1) as

$$\mu=\Pi_{\Omega}[g(\frac{\mu+\mu}{2})-\rho\mathcal{T}(\frac{2w(\mu+\mu)}{2w+\mu+\mu})]. \tag{3.20}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.4).

Algorithm 3.12. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_{\Omega)}[g(\frac{\mu_n+\mu_{n+1}}{2})-\rho\mathcal{T}(\frac{2w(\mu_n+\mu_{n+1})}{2w+\mu_n+\mu_{n+1}})]. \tag{3.21}$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.6 as the predictor and Algorithm 3.12 as corrector. Thus, we obtain a new two-step method for solving the problem (2.4).

Algorithm 3.13. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \Pi_{\Omega}[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})]\\ \mu_{n+1} &= \Pi_{\Omega}\big[g(\frac{\omega_n+\mu_n}{2})-\rho\mathcal{T}\big(\frac{\omega_n+\mu_n}{2}\big)\big],\end{aligned}$$

which is a new predictor-corrector two-step method.

For a parameter ξ, one can rewrite the (3.1) as

$$\mu=\Pi_{\Omega}[g\big((1-\xi)\mu+\xi\mu)\big)-\rho\mathcal{T}(\frac{2w\mu}{w+\mu})].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.4).

Algorithm 3.14. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_{\Omega}[g\big((1-\xi)\mu_n+\xi\mu_{n-1}\big)-\rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})], \quad n=0,1,2,....$$

It is noted that Algorithm 3.14 is equivalent to the following two-step method.

Algorithm 3.15. *For a given μ_0, compute μ_{n+1} by the inertial iterative scheme*

$$\begin{aligned}\omega_n &= (1-\xi)u_n+\xi u_{n-1}\\ \mu_{n+1} &= \Pi_{\Omega}[g(\omega_n)-\rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})].\end{aligned}$$

Using this idea, we can suggest the following iterative methods for solving general harmonic variational inequalities.

Algorithm 3.16. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \xi_n(u_n - u_{n-1}) + \alpha_n \Pi_\Omega[g(\mu_n) - \rho \mathcal{T}(\frac{2w\mu_n}{w+\mu_n})],$$

which is called the inertial proximal point method and appears to be new one.

Here $\alpha_n, \xi_n \geq 0$ are constants and term $\xi_n(u_n - u_{n-1})$ is called the inertial term.

Algorithm 3.17. *For a given u_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\xi)u_n + \xi u_{n-1} \\ u_{n+1} &= \Pi_\Omega[g(y_n) - \rho T(\frac{2wy_n}{w+y_n})], \quad n = 0, 1, 2, ... \end{aligned}$$

We now suggest multi-step inertial methods for solving the general harmonic-like variational inequalities (2.4).

Algorithm 3.18. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}) \\ y_n &= (1-\beta_n)\omega_n + \beta_n \Pi_\Omega\left[g(\frac{\omega_n + \mu_n}{2}) - \rho \mathcal{T}(\frac{\omega_n + \mu_n}{2})\right], \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n \Pi_\Omega\left[g(\frac{\omega_n + y_n}{2}) - \rho \mathcal{T}(\frac{2w(y_n + \omega_n)}{2w + y_n + \omega_n})\right], \end{aligned}$$

where $\beta_n, \alpha_n, \theta_n \in [0,1], \forall n \geq 1$.

Algorithm 3.18 is a three-step modified inertial method for solving general quasi variational inclusion(2.4).

Similarly a four-step inertial method for solving the general harmonic-like variational inequalities (2.4) is suggested.

Algorithm 3.19. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}), \\ t_n &= (1-\gamma_n)\omega_n + \gamma_n \Pi_{(\omega}\left[g(\frac{\omega_n + \mu_n}{2}) - \rho \mathcal{T}\left(\frac{2w(\omega_n + \mu_n)}{2w + \omega_n + \mu_n}\right)\right], \\ y_n &= (1-\beta_n)t_n + \beta \Pi_\Omega\left[g(\frac{t_n + \omega_n}{2}) - \rho \mathcal{T}\left(\frac{2w(t_n + \omega_n)}{2w + \omega_n + t_n}\right)\right], \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n \Pi_\Omega\left[g(\frac{y_n + t_n}{2}) - \rho \mathcal{T}\left(\frac{2w(y_n + t_n)}{2w + y_n + t_n}\right)\right], \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1.$

4 Wiener-Hopf Equations Technique

In this section, we discuss the Wiener-Hopf equations associated with the harmonic-like variational inequalities. It is worth mentioning that the Wiener-Hopf equations associated with variational inequalities were introduced and studied by Shi [60]. For the applications and formulations of the Wiener-Hopf equations, see [36, 52, 53] and the references therein.

We now consider the problem of solving the Wiener-Hopf equations related to the general quasi variational inequalities. Let $\mathcal{T}$ be an operator and $\mathcal{R}_\Omega = \mathcal{I} - \Pi_\Omega$, where $\mathcal{I}$ is the identity operator and Π_Ω is the projection operator.

We consider the problem of finding $z \in \mathcal{H}$ such that

$$\mathcal{T}\Pi_\Omega z + \rho^{-1}\mathcal{R}_\Omega z = 0. \tag{4.1}$$

The equations of the type (4.1) are called the Wiener-Hopf equations. It have been shown that the Wiener-Hopf equations play an important part in the developments of iterative methods, sensitivity analysis and other aspects of the variational inequalities and related optimization problems.

Lemma 4.1. *The element* $\mu \in \Omega$ *is a solution of the harmonic-like variational inequality* (2.4)*, if and only if,* $z \in \mathcal{H}$ *satisfies the resolvent equation* (4.1)*, where*

$$\mu = \Pi_\Omega z, \tag{4.2}$$

$$z = g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu}), \tag{4.3}$$

where $\rho > 0$ *is a constant.*

From Lemma 4.1, it follows that the general harmonic-like variational inequalities (2.4) and the implicit Wiener-Hopf equations (4.1) are equivalent. This alternative formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the harmonic-like variational inequalities and related optimization problems.

We use the Wiener-Hopf equations (4.1) to suggest some new iterative methods for solving the harmonic-like variational inequalities. From (4.2) and (4.3),

$$z = g(\Pi_\Omega z) - \rho\mathcal{T}g(\Pi_\Omega z).$$

Thus, we have

$$g(\mu) = \rho T(\frac{2w\mu}{w+\mu}) + g(\mu) - \rho T g(\Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})])$$

implies that

$$\rho T(\frac{2w\mu}{w+\mu}) - \rho T g(\Pi_\Omega[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})]) = 0.$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n\{\rho T g(\Pi_\Omega)[g(\mu) - \rho T(\frac{2w\mu}{w+\mu})] - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\} \\ &= (1-\alpha_n)\mu + \alpha_n\Pi_\Omega\{\rho\mathcal{T}(\frac{2w\omega}{w+\omega})\omega - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\}, \end{aligned} \tag{4.4}$$

where

$$\omega = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]. \tag{4.5}$$

Using (4.4) and (4.5), we can suggest the following new predictor-corrector method for solving the harmonic-like variational inequalities.

Algorithm 4.1. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Pi_\omega\left\{\rho\mathcal{T}(\frac{2w\omega_n}{w+\omega_n}) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})\right\}. \end{aligned}$$

If $\alpha_n = 1$, then Algorithm 4.1 reduces to

Algorithm 4.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\rho\mathcal{T}(\frac{2w\omega_n}{w+\omega_n}) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})], \end{aligned}$$

which appears to be a new one.

Remark 4.1. *We have only given some glimpse of the technique of the Wiener-Hopf equations for solving the harmonic-like variational inequalities. One can explore the applications of the Wiener-Hopf equations in developing efficient numerical methods for variational inequalities and related nonlinear optimization problems.*

5 Auxiliary Principle Technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et a. [16] as developed in [34,37,41,43–47,51–54,57], to suggest and analyze some proximal point methods for solving general harmonic-like variational inequalities (2.4). We apply the auxiliary principle technique involving an arbitrary operator [34] for finding the approximate solution of the problem (2.4).

For a given $\mu \in \Omega$ satisfying (2.4), find $z \in \Omega$ such that

$$\Big\langle \rho\mathcal{T}\big(\frac{2w(z+\eta(\mu-z))}{w+z+\eta(\mu-z)}\big)+z-g(\mu), \nu - z\Big\rangle + \big\langle M(z)-M(\mu), \nu-z\big\rangle \geq 0, \quad \forall w,\nu \in \Omega, \tag{5.1}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (5.1) is called the auxiliary general harmonic-like variational inequality.

If $z = \mu$, then z is a solution of (2.4). This simple observation enables us to suggest the following iterative method for solving (2.4).

Algorithm 5.1. *For a given $\mu_0 \in$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\Big\langle \rho\mathcal{T}\Big(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))}{2w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1})}\Big)+u_{n+1}-g(\mu), \nu-\mu_{n+1}\Big\rangle + \langle M(\mu_{n+1})-M(\mu_n), \nu-\mu_{n+1}\rangle \geq 0, \quad \forall w,\nu \in \Omega. \tag{5.2}$$

Algorithm 5.1 is called the hybrid proximal point algorithm for solving the general harmonic-like variational inequalities (2.4).

Special Cases: We now discuss some special cases are discussed.

(I). For $\eta = 0$, Algorithm 5.1 reduces to

Algorithm 5.2. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\Big\langle \rho\mathcal{T}\big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\big)+\mu_{n+1}-g(\mu), \nu)-\mu_{n+1}\Big\rangle + \langle M(\mu_{n+1})-M(\mu_n), \nu-\mu_{n+1}\rangle \geq 0, \quad \forall w,\nu \in \Omega, \tag{5.3}$$

is called the implicit iterative methods for solving the problem (2.4).

(II). If $\eta = 1$, then Algorithm 5.1 collapses to

Algorithm 5.3. *For a given $\mu_0 \in \Omega$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho \mathcal{T}(\frac{2w\mu_n}{w+\mu_n}) + \mu_{n+1} - g(\mu), \nu) - \mu_{n+1} \rangle \\ &+ \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1} \rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}$$

is called the explicit iterative method.

(III). For $\eta = \frac{1}{2}$, Algorithm 5.1 becomes:

Algorithm 5.4. *For a given $\mu_0 \in \Omega$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\left\langle \rho \mathcal{T}\left(\frac{2w(\mu_{n+1} + \mu - n)}{2w + \mu_{n+1} + \mu_n}\right) + \mu_{n+1} - \mu, \nu) - \mu_{n+1} \right\rangle \\ &+ \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1} \rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}$$

is known as the mid-point proximal method for solving the problem (2.4).

For the convergence analysis of Algorithm 5.2, we need the following concepts.

Definition 5.1. *An operator $\mathcal{T}$ is said to be harmonic-like pseudomontone with respect to the operator g, if*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) + \mu - g(\mu), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega,$$

implies that

$$-\langle \mathcal{T}(\frac{2w\nu}{w+\nu}) + \nu - g(\mu), \mu - \nu \rangle \geq 0, \quad \forall w, \nu \in \Omega.$$

Theorem 5.1. *Let the operator $\mathcal{T}$ be a harmonic-like pseudo-monotone with respect to the operator g. Let the approximate solution μ_{n+1} obtained in Algorithm 5.2 converges to the exact solution $\mu \in \Omega$ of the problem (2.4). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi \|\mu_{n+1} - \mu_n\| \leq \zeta \|\mu - \mu_n\|. \tag{5.4}$$

Proof. Let $\mu \in \Omega$ be a solution of the problem (2.4). Then,

$$-\langle \rho(\mathcal{T}(\frac{2w\nu}{w+\nu}) + \nu - g(\mu), \mu - \nu \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{5.5}$$

since the operator $\mathcal{T}$ is a harmonic-like pseudo-monotone with respect to the operator g.

Taking $v = \mu_{n+1}$ in (5.5), we obtain

$$-\langle\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) + \mu_{n+1} - g(\mu), \mu - \mu_{n+1})\rangle \geq 0. \tag{5.6}$$

Setting $v = \mu$ in (6.9), we have

$$\begin{aligned}&\langle\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) + \mu_{n+1} - g(\mu), \mu - \mu_{n+1}\rangle \\ &+\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \geq 0.\end{aligned} \tag{5.7}$$

Combining (5.7)and (5.6), we have

$$\begin{aligned}&\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &\geq -\langle\rho\mathcal{T}(\frac{2w(\mu_{n+1}}{w+\mu_{n+1}} + \mu_{n+1} - g(\mu), \mu - \mu_{n+1}\rangle \geq 0.\end{aligned} \tag{5.8}$$

From the equation (5.8), we have

$$\begin{aligned}0 &\leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle,\end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (5.4). □

Theorem 5.2. *Let H be a finite dimensional space and all the assumptions of Theorem 5.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 5.2 converges to the exact solution $\mu \in \Omega$ of (2.4).*

Proof. Let $\mu \in \Omega$ be a solution of (2.4). From (5.4), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi\sum_{n=0}^{\infty}\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \tag{5.9}$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega$. Replacing w_n by μ_{n_j} in (6.9), taking the limit $n_j \longrightarrow \infty$ and using (5.9), we have

$$\langle \rho\mathcal{T}(\frac{2w\hat{\mu}}{w+\hat{\mu}}) + \hat{\mu} - g(\hat{\mu}), \nu - \hat{\mu}\rangle \geq 0, \qquad \forall w, \nu \in \Omega,$$

which implies that $\hat{u}$ solves the problem (2.4) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (\mu_n) = \hat{\mu},$$

which is the required result. □

In recent years, some inertial type iterative methods have been applied to find the approximate solutions of variational inequalities and related optimizations. We again apply the auxiliary approach to suggest some hybrid inertial proximal point schemes for solving the general quasi variational inequalities.

We apply the auxiliary principle technique involving an arbitrary operator for finding the approximate solution of the problem (2.4).

For a given $\mu \in \Omega$ satisfying (2.4), find $z \in \Omega$ such that

$$\begin{aligned}&\left\langle \rho\mathcal{T}(\frac{2w(z+\eta(\mu-z))}{w+z+\eta(\mu-z)}) + z - g(\mu), \nu - z\right\rangle \\ &\quad + \left\langle M(z) - M(\mu) + \alpha_1(\mu-\mu), \nu - z\right\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}\tag{5.10}$$

where $\rho > 0, \eta, \alpha_1 \in [0,1]$ are constants and M is an arbitrary operator. The inequality (5.10) is called the auxiliary general harmonic-like variational inequality.

If $z = \mu$, then z is a solution of (2.4). This simple observation enables us to suggest the following iterative method for solving (2.4).

Algorithm 5.5. *For a given $\mu_0 \in$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\left\langle \rho\mathcal{T}(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))}{2w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1})}) + u_{n+1} - g(\mu), \nu - \mu_{n+1}\right\rangle \\ &\quad + \left\langle M(\mu_{n+1}) - M(\mu_n) + \alpha_1(\mu_n - \mu_{n-1}, \nu - \mu_{n+1}\right\rangle \geq 0, \quad \forall w, \nu \in \Omega. \end{aligned}\tag{5.11}$$

Algorithm 5.6 is called the hybrid proximal point algorithm for solving the general harmonic-like variational inequalities (2.4).

For $\alpha = 0$, Algorithm 5.6 is exactly Algorithm 5.1. Using the technique and ideas of Theorem 5.1 and Theorem 5.2, one can analyze the convergence of Algorithm 5.6 and its special cases.

For $M = I$, the identity operator, Algorithm 5.6 reduces to the following inertial method for solving the problem (2.4).

Algorithm 5.6. *For a given $\mu_0 \in$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\left\langle \rho\mathcal{T}\Big(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))}{2w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1})}\Big)+u_{n+1}-g(\mu),\nu-\mu_{n+1}\right\rangle$$
$$+\left\langle \mu_{n+1}-\mu_n+\alpha_1(\mu_n-\mu_{n-1},\nu-\mu_{n+1}\right\rangle \geq 0, \quad \forall \nu\in\Omega. \tag{5.12}$$

For different and suitable values of the parameters η, α_1, operators $\mathcal{T}, g$ and set-valued convex sets, one can suggest and investigate several new and known methods for solving the general quasi variational inequalities and related nonconvex programming problems. For the implementable numerical methods need further research efforts.

6 Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving quasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [14], which is a first order initial value problem. This implies that the numerical methods for solving initial value and boundary value can be used to develop numerical methods for solving variational inequalities. Consequently, variational inequalities, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. For the applications of dynamical systems, see [19, 25, 37, 40, 42, 45–47, 51–53, 63]. In this section, we consider some iterative methods for solving the general harmonic-like variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \{\Pi_\Omega[g(\mu)-\rho\mathcal{T}(\frac{2w\mu}{w+\mu})]-\mu\}. \tag{6.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu\in\mathcal{H}$ is a solution of the problem(2.4), if and only if, $\mu\in\mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{6.2}$$

We now consider a dynamical system associated with the general quasi variational inequalities (2.4). Using the equivalent formulation (3.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\Big\{\Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \mu\Big\}, \quad \mu(t_0) = \alpha, \tag{6.3}$$

where λ is a parameter. The system of type (6.3) is called the projection dynamical system associated with the problem (2.4). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.4) can be studied.

We note that $\mu \in \Omega$ is a solution of the general quasi variational inequality (2.4), if and only if, $\mu \in \Omega$ is an equilibrium point of the problem 6.3.

Definition 6.1. *[15] The dynamical system is said to converge to the solution set S^* of (6.3), if, irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{6.4}$$

where

$$dist(\mu, S^*) = inf_{\nu\in S^*}\|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (6.4) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 6.2. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq u_1\|\mu(t_0) - \mu^*\| exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 6.1. *(Gronwall Lemma) [14, 25] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now show that the trajectory of the solution of the projection dynamical system (6.3) converges to the unique solution of the general quasi variational inequality (2.4). The analysis is in the spirit of Noor [39] and Xia and Wang [63].

Theorem 6.1. *Let the operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be harmonic-like Lipschitz continuous with constant $\beta > 0$. Let the operator g be Lipschitz continuous with constant $\zeta > 0$. If $\lambda\{(1+\eta+\zeta+\rho\beta\} < 1$ and Assumption 2.1 then, for each $\mu_0 \in \Omega$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (6.3) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \mu\}, \quad \forall w, \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$, For $\forall \mu, \nu \in H$, we have

$$\begin{aligned} \|G(\mu) - G(\nu)\| &\leq \lambda\left\{\Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\| + \|\mu - \nu\|\right\} \\ &\leq \lambda\{\|\mu - \nu\| + \|g(\mu) - g(\nu) - \rho(\mathcal{T}\mu - \mathcal{T}\nu)\} \\ &\leq \lambda\{\|\mu - \nu\| + \{\zeta + \rho\beta\}\|\mu - \nu\|\} \\ &\leq \lambda\{(1 + \zeta + \rho\beta\}\|\mu - \nu\| \end{aligned}$$

where we have used the fact that g is Lipschitz continuous with a constant ζ and the operator $\mathcal{T}$ is harmonic-like Lipschitz continuous with constant $\beta > 0$, respectively. This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(1+\zeta+\rho\beta\} < 1$ and for each $\mu \in \Omega$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (6.3), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that

$T_1 = \infty$. Consider, for any $\mu \in \Omega$,

$$\begin{aligned}
\|G(\mu)\| = \|\frac{d\mu}{dt}\| &= \lambda\|[g(u) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \mu\| \\
&\leq \lambda\{\|\Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \Pi_\Omega[0]\| + \|\Pi_\Omega[0] - \mu\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho T(\frac{2w\mu}{w+\mu})\| + \|\Pi_\Omega[g(\mu)] - \Pi_\Omega[0]\| + \|\Pi_\Omega[0] - \mu\|\} \\
&\leq \lambda\{(\rho\beta + 2 + \zeta)\|\mu\| + \|\Pi_\Omega[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\|ds \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\|ds,
\end{aligned}$$

where $k_1 = \lambda\|\Pi_\Omega[0]\|$ and $k_2 = \rho\beta + 2 + \zeta$. Hence by the Gronwall Lemma 6.1, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 6.2. *Let the operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be harmonic-like Lipschits continuous with constant β. If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous $\zeta > 0$, then the dynamical system (6.3) converges globally exponentially to the unique solution of the general harmonic-like variational inequality (2.4).*

Proof. Since the operator g is Lipschitz continuous, it follows from Theorem 6.1 that the dynamical system (6.3) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (6.3). For a given $\mu^* \in H$ satisfying (2.4), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \Omega. \tag{6.5}$$

From (6.3) and (6.5), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\langle\mu(t)-\mu^*, \frac{d\mu}{dt}\rangle \\
&= 2\lambda\langle\mu(t)-\mu^*, \Pi_\Omega[g(\mu(t))-\rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})]-\mu(t)\rangle \\
&= 2\lambda\langle\mu(t)-\mu^*, \Pi_\Omega[g(\mu(t))-\rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})]-\mu^*(t)+\mu^*(t)-\mu(t)\rangle \\
&= -2\lambda\langle\mu(t)-\mu^*(t), \mu(t)-\mu^*(t)\rangle \\
&\quad +2\lambda\langle\mu(t)-\mu^*(t), \Pi_{\Omega(}[g(\mu(t))-\rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})-\mu^*(t)\rangle \\
&\le -2\lambda\langle\rho(\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})-\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)}), g(\mu(t))-g(\mu^*(t))\rangle \\
&\quad +2\lambda\langle\mu(t)-\mu^*, \Pi_\Omega[g(\mu(t))-\rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})]-\Pi_\Omega[g(\mu^*(t))-\rho\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)})]\rangle, \\
&\le -2\lambda\sigma\|\mu(t)-\mu^*\|^2+\lambda\|\mu(t)-\mu^*\|^2 \\
&\quad +\lambda\|\Pi_\Omega[g(\mu(t))-\rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})]-\Pi_\Omega[g(\mu^*(t))-\rho\mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)})]\|^2 \qquad (6.6)
\end{aligned}$$

Using the Lipschitz continuity of the operators $\mathcal{T}, g$, we have

$$\begin{aligned}
&\|\Pi_\Omega[g(\mu)-\rho\mathcal{T}(\frac{2w\mu}{w+\mu})\mu]-\Pi_\Omega[g(\mu^*)-\rho\mathcal{T}(\frac{2w\mu^*}{w+\mu^*})]\| \\
&\le \delta\|g(\mu)-g(\mu^*)-\rho(\mathcal{T}(\frac{2w\mu}{w+\mu})\mu-\mathcal{T}(\frac{2w\mu^*}{w+\mu^*}))\| \\
&\le \delta(\zeta+\rho\beta)\|\mu-\mu^*\|. \qquad (6.7)
\end{aligned}$$

From (6.6) and (6.7), we have

$$\frac{d}{dt}\|\mu(t)-\mu^*\| \le 2\xi\lambda\|\mu(t)-\mu^*\|,$$

where

$$\xi = (\delta(\zeta+\rho\beta)-1).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t)-\mu^*\| \le \|\mu(t_0)-\mu^*\|e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (6.3) converges globally exponentially to the unique solution of the general harmonic-like variational inequality (2.4). □

We use the projection dynamical system (6.3) to suggest some iterative for solving the harmonic-like variational inequalities (2.4). These methods can be viewed in the sense of Korpelevich [21] and Noor [36, 37] involving the double projections.

For simplicity, we take $\lambda = 1$. Thus the dynamical system(6.3) becomes

$$\frac{d\mu}{dt} + \mu = \Pi_\Omega\big[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\big], \quad \mu(t_0) = \alpha. \tag{6.8}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (6.8), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_n = \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \tag{6.9}$$

where h is the step size.

Now, we can suggest the following implicit iterative method for solving the general harmonic-like variational inequality (2.4).

Algorithm 6.1. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_\Omega\big[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) - \frac{\mu_{n+1} - \mu_n}{h}\big].$$

This is an implicit method. Algorithm 6.1 is equivalent to the following two-step method.

Algorithm 6.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_\Omega[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \Pi_\Omega\big[g(\mu_n) - \rho\mathcal{T}(\frac{2w\omega_n}{w+\omega_n}) - \frac{\omega_n - \mu_n}{h}\big]. \end{aligned}$$

Remark 6.1. *For appropriate and suitable choice of the operators $\mathcal{T}, g$, convex set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving general harmonic-like variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al [52, 53], one can discuss the convergence analysis of the proposed methods.*

We use the second order dynamical system to suggest and investigate some inertial proximal methods for solving the general harmonic-like variational inequalities (2.4). These inertial implicit methods are constructed using the central finite difference schemes and its variant forms.

To be more precise, we consider the problem of finding $\mu \in \Omega$ such that

$$\gamma\frac{d^2\mu}{d^2x} + \frac{d\mu}{dx} + \mu = \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})], \quad \mu(a) = \alpha, \mu(b) = \beta, \tag{6.10}$$

where $\gamma \geq 0$, $\quad \eta \geq 0$ and $\rho > 0$ are constants. Problem (6.10) is called second order dynamical system. This is a second boundary value problem associated with the problem (2.4).

We discretize the second-order dynamical systems (6.10) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\eta\frac{\mu_n-\mu_{n-1}}{h}+\mu_n=\Pi_\Omega[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \tag{6.11}$$

where h is the step size.

If $\gamma=1, h=1, \eta=1$ then, from equation(6.11) we have

Algorithm 6.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_\Omega[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \quad n=0,1,2,\ldots.$$

which is the extragradient method of Korpelevich [21] type for solving the harmonic-like variational inequalities.

Algorithm 6.3 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the two-step inertial method.

Algorithm 6.4. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta_n\mu_{n-1} \\ \mu_{n+1} &= \Pi_\Omega[g(\mu_n)-\rho\mathcal{T}(\frac{2wy_n}{w+y_n})], \quad n=0,1,2,\ldots \end{aligned}$$

where $\theta_n\in[0.1]$ is a constant.

Similarly, we suggest the following iterative method.

Algorithm 6.5. *For given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_\Omega[g(\mu_{n+1})-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \quad n=0,1,2,\ldots.$$

which is known as the double projection method and can be written as

Algorithm 6.6. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta\mu_{n-1} \\ \mu_{n+1} &= \Pi_\Omega\big[g(y_n)-\rho\mathcal{T}(\frac{2wy_n}{w+y_n})\big], \quad n=0,1,2,\ldots. \end{aligned}$$

which is called the two-step inertial iterative method.

We discretize the second-order dynamical systems (6.3) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+\mu_{n+1}=\Pi_\Omega\big[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})\big],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the general harmonic-like variational inequalities (2.4).

Algorithm 6.7. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_\Omega\Big[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})-\frac{\gamma\mu_{n+1}-(2\gamma-h)\mu_n+(\gamma-h)\mu_{n-1}}{h^2}\Big].$$

Algorithm 6.7 is called the inertial proximal method for solving the harmonic-like variational inequalities and related optimization problems. This is a new proposed method.

We note that, for $\gamma = 0$, Algorithm 6.7 reduces to the following iterative method for solving harmonic-like variational inequalities (2.4).

Algorithm 6.8. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_\Omega\Big[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})-\frac{\mu_n-\mu_{n-1}}{h}\Big],\quad n=0,1,2,\ldots.$$

We again discretize the second-order dynamical systems (6.10) using central difference scheme and forward difference scheme to suggest the following inertial proximal method for solving (2.4).

Algorithm 6.9. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_\Omega\Big[g(\mu_{n+1})-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})-\frac{(\gamma+h)\mu_{n+1}-(2\gamma+h)\mu_n+\gamma\mu_{n-1}}{h^2}\Big].$$

Algorithm 6.9 is quite different from other inertial proximal methods for solving the quasi variational inequalities.

If $\gamma = 0$, then Algorithm 6.9 collapses to:

Algorithm 6.10. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_\Omega\Big[g(\mu_{n+1})-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})-\frac{\mu_{n+1}-\mu_n}{h}\Big].$$

Algorithm 6.9 is an proximal method for solving the harmonic-like variational inequalities. Applying the technique and ideas of Noor et al. [52,53], one can study the convergence criteria of these Algorithms with some modifications and adjustment. Such type of proximal methods were suggested by Noor [36] using the fixed point problems. In brief, by suitable discretization of the second-order dynamical systems (6.10), one can construct a wide class of explicit and implicit method for solving quasi variational inequalities and their variant forms.

7 Change of Variable Method

In this section, we consider the change of variable method for solving general harmonic-like variational inequalities (2.4). This technique is mainly due to Noor [33] and Al-Said et al. [1]. For the sake of completeness and to convey the main idea, we include some details.

We note that the complementarity problem (2.8) can be rewritten in the following form:

$$w = g(\mu) \in \Omega, \quad \nu = \mathcal{T}(\frac{w\mu}{w+\mu}) \in \Omega^*, \quad \langle \mathcal{T}(\frac{w\mu}{w+\mu}), g(\mu)\rangle = 0 \tag{7.1}$$

which is useful in developing a fixed point formulation.

It is well known that, for $z \in \mathcal{H}$, we have

$$z = \Pi_\Omega z + \Pi_{-\Omega^*} z = \Pi_\Omega z + \Pi_{\Omega^*}(-z). \tag{7.2}$$

Following the idea of Al-Said and Noor [1], we consider the following change of variables

$$g(\mu) = \frac{|z| + z}{2} = z^+ = \Pi_\Omega(z) \tag{7.3}$$

and

$$\nu = \frac{|z| - z}{2\rho} = \rho^{-1} z^-. \tag{7.4}$$

From (7.2), (7.3) and (7.4), we have

$$\mu = g^{-1}\Pi_\Omega z \tag{7.5}$$

$$\begin{aligned} z &= z^+ - z^- = \Pi_\Omega z + \Pi_\omega(-z) \\ &= g(\mu) - \rho\mathcal{T}g^{-1}\Pi_\Omega z = g(\mu) - \rho\mathcal{T}(\frac{w\mu}{w+\mu}). \end{aligned} \tag{7.6}$$

Combining (7.5) and (7.6), we obtain

$$\mu = \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{w\mu}{w+\mu})]. \tag{7.7}$$

Thus, we have shown that the harmonic-like complementarity problem (2.8) is equivalent to the fixed point problem (7.7). This implies that $\mu \in \Omega$ is the solution of the general harmonic-like variational inequality (2.4). That is, $\mu \in \Omega$ satisfies the inequality

$$\langle \mathcal{T}(\frac{w\mu}{w+\mu}), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega.$$

This can be rewritten equivalently as finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}(\frac{w\mu}{w+\mu}) + \mu - g(\mu), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega,$$

which is the general harmonic variational inequality (2.4).

In recent years, this technique have been used to develop modulus based methods for solving the system of absolute value equations, which is another area in numerical analysis and optimization. This approach can be extended for solving the mixed variational inequalities, which needs further research efforts.

8 Applications and Future Research

We would like to mention that some of the results obtained and presented in this paper can be extended for more general harmonic-like variational inequalities. To be more precise, let $C(H)$ be a family of nonempty compact subsets of H. Let $T, h, g, f : H \longrightarrow C(H)$ be the operators. We consider the problem of finding$\mu \in \Omega$ such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu} + h(\mu) - g(\nu), f(\nu) - h(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{8.1}$$

which is called the general harmonic-like variational inequality. We would like to mention that one can obtain various classes of general harmonic-like variational inequalities for appropriate and suitable choices of the the operators $\mathcal{T}, g, h$, and convex set Ω.

Note that, if $w = \mu, \quad h = I, f = I$, then the problem (8.1) is equivalent to find $\mu \in \Omega$, such that

$$\langle Tu + \mu - g(\mu), \nu) - \mu \rangle \geq 0 \quad \forall \nu \in \Omega,$$

which is the problem(2.4).

Using Lemma 3.1, one can prove that the problem (8.1) is equivalent to finding $\mu \in \Omega$ such that

$$h(u) = \Pi_{\Omega}[g(\mu) - \rho \mathcal{T}(\frac{2w\mu}{w+\mu})] \tag{8.2}$$

which can be written as

$$\mu = \mu - h(\mu) + \Pi_\Omega[g(\mu) - \rho].\mathcal{T}(\frac{2w\mu}{w+\mu})].$$

Thus one can consider the mapping F associated with the problem (8.1) as

$$F(\mu) = \mu - h(\mu) + \Pi_\Omega[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})],$$

which can be used to discuss the uniqueness of the solution of the problem (8.1).

From (8.1) and (8.2, it follows that the general harmonic-like variational inequalities are equivalent to the fixed problems. Consequently, all the results obtained for the problem (2.4) continue to hold for the problem (8.1) with suitable modifications and adjustments. The development of efficient implementable numerical methods for solving the general harmonic-like variational inequalities and non optimization problems requires further efforts. Despite the research activates, very few numerical results are available. The development of efficient implementable numerical methods for solving the general harmonic-like variational inequalities and non optimizations problems requires further efforts.

Conclusion

In this paper, we have introduced and studied some new classes of general harmonic-like variational inequalities for two arbitrary operators. By inter changing the roles of these operators, one can obtain a number of new classes of harmonic-like variational inequalities and complementarity problems. Applying the projection technique, we have established the equivalence between the fixed point problems and harmonic-like variational inequalities. This equivalence formulation is used to study the unique existence of solution. Several hybrid multi-step iterative methods for solving the harmonic-like variational inequalities are suggested applying the fixed point, the Wiener-Hopf equations and dynamical systems. These new methods include extragradient method, modified double projection methods and inertial type methods. Convergence analysis of the proposed method is discussed under suitable weaker conditions. Change of variable is used to establish the equivalence between the harmonic-like complementarity problems, which can be used for solving the harmonic-like variational inequalities. This is a new novel approach. It is an open problem to compare these proposed methods with other methods. We have shown that the general harmonic variational inequalities are equivalent to the extended general variational inequalities with suitable conditions of the convex set. Applications of the fuzzy set theory, stochastic, quantum calculus, fractal, fractional and random can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities. One may explore these aspects of the general harmonic-like variational inequality and its variant forms in these areas. Using the ideas and techniques of this paper, one can explore the applications of these methods for solving the complementarity problems and related mathematical programming problems.

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