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F-Alternate Interpolative Ciric-Reich-Rus Contraction Mapping Theorem

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Abstract

In [1], the authors introduced the interpolative Ciric-Reich-Rus operator in Branciari metric space and obtained some fixed point theorems. In [2], an alternate characterization of the interpolative Ciric-Reich-Rus operator was given, and some fixed point theorems were obtained. In the present paper, we consider the alternate interpolative Ciric-Reich-Rus operator is an F-contraction [3], and obtain a fixed point theorem.

1 Introduction and Preliminaries

Notation 1.1. *[3] Ξ will denote the class of all mappings $F : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying*

(a) *F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.*

(b) *For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

(c) *There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Example 1.2. *[3] The following are elements of Ξ*

(a) $F(\alpha) = ln(\alpha)$.

(b) $F(\alpha) = ln(\alpha) + \alpha$.

(c) $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$, *where* $\alpha > 0$.

(d) $F(\alpha) = ln(\alpha^2 + \alpha)$, *where* $\alpha > 0$.

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Definition 1.3. *[3] Let (X,d) be a metric space. A mapping $T : X \mapsto X$ is said to be an F-contraction if there exists $\tau > 0$ and $F \in \Xi$ such that*

$$d(Tx,Ty) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)).$$

Theorem 1.4. *[3] Let (X,d) be a complete metric space, and let $T : X \mapsto X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Definition 1.5. *[1] Let (X,d) be a Branciari distance space. A self-mapping T on X is called an interpolative Ciric-Reich-Rus type contraction, if there are $\lambda \in [0,1)$ and positive reals α, β with $\alpha+\beta < 1$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\beta} d(x,Tx)^{\alpha} d(y,Ty)^{1-\alpha-\beta}$$

for all $x,y \in X \backslash Fix(T)$.

Theorem 1.6. *[1] Let $T : X \mapsto X$ be an interpolative Ciric-Reich-Rus type contraction on a complete Branciari distance space (X,d). Then T has a fixed point in X.*

Definition 1.7. *[2] Let (X,d) be a metric space. A map $T : X \mapsto X$ is called an alternate Ciric-Reich-Rus operator if there exists $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\frac{1}{3}} d(x,Tx)^{\frac{1}{3}} d(y,Ty)^{\frac{1}{3}}$$

for all $x,y \in X \backslash Fix(T)$.

Theorem 1.8. *[2] Let (X,d) be a metric space. Suppose $T : X \mapsto X$ is an alternate interpolative Ciric-Reich-Rus operator, that is, there exits $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\frac{1}{3}} d(x,Tx)^{\frac{1}{3}} d(y,Ty)^{\frac{1}{3}}$$

for all $x,y \in X \backslash Fix(T)$. If X is complete, then the fixed point exists.

Definition 1.9. *[4] Let (X,d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\alpha} d(x,Tx)^{1-\alpha}$$

where $\lambda \in [0,1)$ and $\alpha \in (0,1)$, for all $x,y \in X$, $x,y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows

Definition 1.10. *[4] Let (X,d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\frac{1}{2}} d(x,Tx)^{\frac{1}{2}}$$

where $\lambda \in [0,1)$, for all $x,y \in X$, $x,y \notin Fix(T)$.

Theorem 1.11. *[4] Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator. If (X, d) is complete, then the fixed point of T exists.*

Definition 1.12. *[5] Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is called an F-interpolative Berinde weak contraction if there exists $\tau > 0$ and $F \in \Xi$ such that*

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}).$$

Theorem 1.13. *[5] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a continuous F-interpolative Berinde weak contraction. Then T has a fixed point $x^* \in X$, and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

2 Main Result

Definition 2.1. *Let (X, d) be a metric space. A mapping $T : X \mapsto X$ will be called an F-alternate interpolative Ciric-Reich-Rus contraction if there exists $\tau > 0$ and $F \in \Xi$ such that*

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F\left(\frac{1}{2^{\frac{1}{3}}} d(x, y)^{\frac{1}{3}} d(x, Tx)^{\frac{1}{3}} d(y, Ty)^{\frac{1}{3}}\right).$$

Example 2.2. *Let $F : \mathbb{R}_+ \mapsto \mathbb{R}$ be given by the formula $F(\alpha) = ln(\alpha)$. Observe that F satisfies (a), (b), and (c) of Notation 1.1. for any $\lambda \in (0, 1)$. Each mapping $T : X \mapsto X$ satisfying the implication in the previous definition is an F-alternate interpolative Ciric-Reich-Rus contraction such that*

$$d(Tx, Ty) \leq \frac{e^{-\tau}}{2^{\frac{1}{3}}} d(x, y)^{\frac{1}{3}} d(x, Tx)^{\frac{1}{3}} d(y, Ty)^{\frac{1}{3}}$$

for all $x, y \in X \backslash Fix(T)$, $Tx \neq Ty$. Note that for all $x, y \in X \backslash Fix(T)$, such that $Tx = Ty$, the inequality

$$d(Tx, Ty) \leq \frac{e^{-\tau}}{2^{\frac{1}{3}}} d(x, y)^{\frac{1}{3}} d(x, Tx)^{\frac{1}{3}} d(y, Ty)^{\frac{1}{3}}$$

still holds, that is, T is an alternate interpolative Ciric-Reich-Rus contraction [2].

Theorem 2.3. *Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a continuous F-alternate interpolative Ciric-Reich-Rus contraction. Then T has a fixed point $x^* \in X$, and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. Define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Denote $\gamma_n = d(x_n, x_{n+1})$ for $n = 0, 1, 2, \cdots$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T, and the proof is finished. So we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then $\gamma_n > 0$ for all $n \in \mathbb{N}$. Since T is an F-alternate interpolative Ciric-Reich-Rus contraction, we deduce the following

$$\begin{aligned}
\tau + F(\gamma_n) &= \tau + F(d(Tx_{n-1}, Tx_n)) \\
&\leq F\left(\frac{1}{2^{\frac{1}{3}}} d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{3}} d(x_n, Tx_n)^{\frac{1}{3}}\right) \\
&\leq F\left(\frac{1}{2^{\frac{1}{3}}} d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}}\right) \\
&\leq F\left(\frac{1}{2^{\frac{1}{3}}} d(x_{n-1}, x_n)^{\frac{2}{3}} (d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1}))^{\frac{1}{3}}\right) \\
&\leq F\left(\frac{1}{2^{\frac{1}{3}}} d(x_{n-1}, x_n)^{\frac{2}{3}} (2d(x_n, x_{n-1}))^{\frac{1}{3}}\right) \\
&\leq F(d(x_{n-1}, x_n)) \\
&\leq F(\gamma_{n-1})
\end{aligned}$$

which implies

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau.$$

The above implies that $\lim_{n\to\infty} F(\gamma_n) = -\infty$. It now follows from (b) of Notation 1.1, that $\lim_{n\to\infty} \gamma_n = 0$. From (c) of Notation 1.1, there exists $\lambda \in (0, 1)$ such that $\lim_{n\to\infty} \gamma_n^\lambda F(\gamma_n) = 0$. Since

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau$$

we deduce the following

$$\begin{aligned}
\gamma_n^\lambda F(\gamma_n) - \gamma_n^\lambda F(\gamma_0) &\leq \gamma_n^\lambda (F(\gamma_0) - n\tau) - \gamma_n^\lambda F(\gamma_0) \\
&= -\gamma_n^\lambda n\tau \\
&\leq 0.
\end{aligned}$$

Since $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} \gamma_n^\lambda F(\gamma_n) = 0$. If we take limits in the above inequality, we deduce that $\lim_{n\to\infty} n\gamma_n^\lambda = 0$. This suggests that there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^\lambda \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\gamma_n \leq \frac{1}{n^{\frac{1}{\lambda}}}$$

for all $n \geq n_1$. Now we show that $\{x_n\}$ is Cauchy. Consider, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. From the definition of the metric and the above inequality we get

$$\begin{aligned}
d(x_m, x_n) &\leq \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n \\
&\leq \sum_{i=n}^{\infty} \gamma_i \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}.
\end{aligned}$$

Since $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}$ is convergent, it follows from the above inequality that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Finally, since T is continuous, we deduce the following

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

which implies that x^* is a fixed point of T, and the proof is finished. □

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