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Pseudosymmetric Characterizations for Sasakian Manifolds Admitting General Connection

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Abstract

In this study, the geometry of Sasakian manifolds is investigated using a general connection instead of the classical Levi-Civita connection. On a Sasakian manifold admitting a general connection, we first define the projective and concircular curvature tensors and obtain the characterizations of projectively flat, concircularly flat, projectively semi-symmetric, and concircularly semi-symmetric Sasakian manifolds. Additionally, by discuss the act of projective and concircular curvature tensors on each other, we reveal the properties of Sasakian manifolds admitting a general connection. In the next section, we construct Ricci-Bourguignon solitons on Sasakian manifolds admitting a general connection. In this connection, we search Ricci pseudo-symmetric, projectively Ricci pseudo-symmetric, and concircularly Ricci pseudo-symmetric Sasakian manifolds admitting Ricci-Bourguignon solitons. Consequently, we compare all these important properties on Sasakian manifolds separately according to the Tanaka-Webster, Schouten-van Kampen, and Zamkovoy connections.

1 Introduction

Sasakian manifolds are an important class in differential geometry and especially in Riemannian geometry. Sasakian manifolds can be thought of as Riemannian manifolds with a 1-dimensional contact structure and can be viewed as a strange but natural generalization of Kaehler manifolds. These manifolds have important applications in mathematics, physics, and engineering. Because it is related to contact geometry, it is used for analysis of mechanical systems such as robot arms and dynamic control systems. Especially in areas such as contact mechanical systems and automatic motion planning, Sasakian geometry can be used. In geometric optics models, Sasakian manifolds can be used to understand how light travels in curved spaces. Particle motion under magnetic fields can be modeled using contact and Sasakian

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structures. Sasakian manifolds, which have applications in many similar areas, are a very important classes for differential geometry.

A connection on a manifold provides a way to differentiate vector fields along curves. More formally, a connection allows the definition of a derivative of a vector field along another vector field, facilitating the study of how vectors change in a manifold's curved geometry. Levi-Civita connection is the most common type of connection, uniquely determined for a Riemannian manifold. It is compatible with the metric and is torsion-free, meaning the connection does not introduce any twisting in the vectors.

General connection, often referred to as a connection on a differentiable manifold, is a fundamental concept in differential geometry and plays a crucial role in the study of curved spaces. General connections are a powerful tool in understanding the geometric structure of manifolds. They provide the framework for defining differentiation in curved spaces and have significant implications in both mathematics and physics. The study of connections continues to be an active area of research, leading to deeper insights into the geometry and topology of manifolds.

In this study, the geometry of Sasakian manifolds is investigated using a general connection instead of the classical Levi-Civita connection. On a Sasakian manifold admitting a general connection, we first define the projective and concircular curvature tensors and obtain the characterizations of projectively flat, concircularly flat, projectively semi-symmetric, and concircularly semi-symmetric Sasakian manifolds. Additionally, by discuss the act of projective and concircular curvature tensors on each other, we reveal the properties of Sasakian manifolds admitting a general connection. In the next section, we construct Ricci-Bourguignon solitons on Sasakian manifolds admitting a general connection. In this connection, we search Ricci pseudo-symmetric, projectively Ricci pseudo-symmetric, and concircularly Ricci pseudo-symmetric Sasakian manifolds admitting Ricci-Bourguignon solitons. Consequently, we compare all these important properties on Sasakian manifolds separately according to the Tanaka-Webster, Schouten-van Kampen, and Zamkovoy connections.

2 Preliminary

An almost contact structure on a smooth manifold M of dimension $n = (2m+1)$ is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, and η is a 1-form on M satisfying

$$\phi^2\Theta_1 = -\Theta_1 + \eta(\Theta_1)\xi, \quad \eta(\xi) = 1, \tag{1}$$

$$\eta(\phi\Theta_1) = 0, \quad \phi\xi = 0, \quad rank\phi = 2n. \tag{2}$$

A smooth manifold M endowed with an almost contact structure is called an almost contact manifold.

A Riemannian metric g on M is said to be compatible with an almost contact structure (ϕ, ξ, η), if

$$g(\phi\Theta_1, \phi\Theta_2) = g(\Theta_1, \Theta_2) - \eta(\Theta_1)\eta(\Theta_2), \tag{3}$$

for all $\Theta_1, \Theta_2 \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all vector fields on M. An almost contact manifold endowed with a compatible Riemannian metric is said to be an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g). Furthermore, the fundamental 2-form Φ on $M(, \phi, \xi, \eta, g)$ is defined by

$$\Phi(\Theta_1, \Theta_2) = g(\Theta_1, \phi\Theta_2) \tag{4}$$

for all $\Theta_1, \Theta_2 \in \chi(M)$. An almost contact metric manifold is said to be Sasakian manifold if

$$(D_{\Theta_1}\phi)\Theta_2 = g(\Theta_1, \Theta_2)\xi - \eta(\Theta_2)\Theta_1, \tag{5}$$

where D denotes Levi-Civita connection admitting the Riemannian connection of g. From (5), we conclude that for a Sasakian structure

$$D_{\Theta_1}\xi = -\phi\Theta_1. \tag{6}$$

We have the following Lemma for later use.

Lemma 1. *n-dimensional Sasakian manifold the following relation holds:*

$$R(\Theta_1, \Theta_2)\xi = \eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \tag{7}$$

$$R(\xi, \Theta_2)\Theta_3 = g(\Theta_2, \Theta_3)\xi - \eta(\Theta_3)\Theta_2, \tag{8}$$

$$R(\Theta_1, \xi)\Theta_3 = -g(\Theta_1, \Theta_3)\xi + \eta(\Theta_3)\Theta_1, \tag{9}$$

$$S(\Theta_1, \xi) = (n-1)\eta(\Theta_1), \tag{10}$$

$$(D_{\Theta_1}\eta)\Theta_2 = g(\Theta_1, \phi\Theta_2), \tag{11}$$

$$Q\xi = (n-1)\xi, \tag{12}$$

for all vector fields Θ_1, Θ_2 and Θ_3 on M, where R, S and Q are Riemann curvature tensor,Ricci tensor and Ricci operator respectively.

It is also defined as

$$S(\Theta_1, \Theta_2) = g(Q\Theta_1, \Theta_2).$$

In this paper, the symbols D^G, D, D^q, D^Z, D^S and D^T are, respectively, denoted for general connection, Levi-Civita connection, quarter-symmetric metric connection, Zamkovoy connection, Schouten-Van Kampen connection and generalized Tanaka-Webster connection, respectively.

Recently, Biswas and Baishya introduced and studied a new connection, named general connection in Sasakian geometry [1, 2]. In the contact geometry, the general connection D^G is defined as

$$D^G_{\Theta_1}\Theta_2 = D^{\Theta_2}_{\Theta_1} + \kappa_1 \left[(D_{\Theta_1}\eta)(\Theta_2)\xi - \eta(\Theta_2) D_{\Theta_1}\xi \right] + \kappa_2 \eta(\Theta_1)\phi\Theta_2, \tag{13}$$

the pair (κ_1, κ_2) being real constants. The beauty of such connection D^G lies in the fact that it has the flavour of

· quarter symmetric metric connection for $(\kappa_1, \kappa_2) = (0, -1)$ in [3, 4],

· Schouten-Van Kampen connection for $(\kappa_1, \kappa_2) = (1, 0)$ in [5],

· Tanaka Webster connection for $(\kappa_1, \kappa_2) = (1, -1)$ in [6],

· Zamkovoy connection for $(\kappa_1, \kappa_2) = (1, 1)$ in [7].

The torsion tensor T of the connection D^G in Sasakian manifold satisfies

$$\begin{aligned} T^G(\Theta_1, \Theta_2) &= 2\kappa_1 g(\Theta_1, \phi\Theta_2)\xi + \kappa_1 \left[\eta(\Theta_2)\phi\Theta_1 - \eta(\Theta_1)\phi\Theta_2 \right] \\ &+ \kappa_2 \left[\eta(\Theta_1)\phi\Theta_2 - \eta(\Theta_2)\phi\Theta_1 \right]. \end{aligned} \tag{14}$$

If we choose $\Theta_2 = \xi$ in (13) and by using (6), we have

$$D^G_{\Theta_1}\xi = (\kappa_1 - 1)\phi\Theta_1. \tag{15}$$

Lemma 2. *For an n-dimensional Sasakian manifold admitting general connection and if R^G, S^G, r^G, Q^G are Riemannian curvature tensor, Ricci tensor,scalar curvature and Ricci operator in general connection, then following results ([1, 2]) hold:*

$$\begin{aligned} R^G(\Theta_1, \Theta_2)\Theta_3 &= R(\Theta_1, \Theta_2)\Theta_3 + \left(\kappa_1^2 - 2\kappa_1\right) \left[g(\Theta_3, \phi\Theta_1)\phi\Theta_2 + g(\Theta_2, \phi\Theta_3)\phi\Theta_1 \right] \\ &+ (\kappa_1 - \kappa_1\kappa_2 + \kappa_2) \left[g(\Theta_1, \Theta_3)\eta(\Theta_2)\xi - g(\Theta_2, \Theta_3)\eta(\Theta_1)\xi \right. \\ &\left. + \eta(\Theta_1)\eta(\Theta_3)\Theta_2 - \eta(\Theta_2)\eta(\Theta_3)\Theta_1 \right] - 2\kappa_2 g(\Theta_2, \phi\Theta_1)\phi\Theta_3, \end{aligned} \tag{16}$$

$$S^G(\Theta_2,\Theta_3)=S(\Theta_2,\Theta_3)-\Lambda_1 g(\Theta_2,\Theta_3)+\Lambda_2\eta(\Theta_2)\eta(\Theta_3), \tag{17}$$

$$S^G(\Theta_2,\xi)=-S^G(\xi,\Theta_2)=-(n-1)\Lambda_3\eta(\Theta_2), \tag{18}$$

$$Q^G\Theta_2=Q\Theta_2-\Lambda_1\Theta_2+\Lambda_2\eta(\Theta_2)\xi, \tag{19}$$

$$Q^G\xi=-(n-1)\Lambda_3\xi, \tag{20}$$

$$r^G=r-\Lambda_1 n+\Lambda_2, \tag{21}$$

$$R^G(\Theta_1,\Theta_2)\xi=\Lambda_3[\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1], \tag{22}$$

$$R^G(\xi,\Theta_2)\Theta_3=\Lambda_3[-g(\Theta_2,\Theta_3)\xi+\eta(\Theta_3)\Theta_2], \tag{23}$$

$$R^G(\Theta_1,\xi)\Theta_3=\Lambda_3[g(\Theta_1,\Theta_3)\xi-\eta(\Theta_3)\Theta_1], \tag{24}$$

and where

$$\Lambda_1=\kappa_1^2-\kappa_1-\kappa_2-\kappa_1\kappa_2, \tag{25}$$

$$\Lambda_2=\kappa_1^2+(n-2)\kappa_1\kappa_2-n(\kappa_1+\kappa_2), \tag{26}$$

$$\Lambda_3=\kappa_1-\kappa_1\kappa_2+\kappa_2-1, \tag{27}$$

for all $\Theta_1,\Theta_2,\Theta_3\in\chi(M)$.

If we make calculations according to the general connection, some other connections can be expressed as follows with the help of some special choices of $\Lambda_1, \Lambda_2, \Lambda_3$:

- For quarter-symmetric metric connection

$$\Lambda_1 = 1, \Lambda_2 = n, \Lambda_3 = -2, \tag{28}$$

- For generalized Tanaka-Webster connection

$$\Lambda_1 = 2, \Lambda_2 = 3 - n, \Lambda_3 = 0, \tag{29}$$

- For Zamkovoy connection

$$\Lambda_1 = -2, \Lambda_2 = -1 - n, \Lambda_3 = 0, \tag{30}$$

- For Schouten-Van Kampen connection

$$\Lambda_1 = 0, \Lambda_2 = 1 - n, \Lambda_3 = 0. \tag{31}$$

Lemma 3. *The following relations hold for the projective curvature tensor defined on an $n = (2m+1)$-dimensional Sasakian manifold admitting a general connection.*

$$P^G(\Theta_1, \Theta_2)\Theta_3 = R^G(\Theta_1, \Theta_2)\Theta_3 - \frac{1}{n-1}\left[S^G(\Theta_2, \Theta_3)\Theta_1 - S^G(\Theta_1, \Theta_3)\Theta_2\right], \tag{32}$$

$$P^G(\xi, \Theta_2)\Theta_3 = -\Lambda_3 g(\Theta_2, \Theta_3)\xi - \frac{1}{n-1}S^G(\Theta_2, \Theta_3), \tag{33}$$

$$P^G(\Theta_1, \xi)\Theta_3 = \Lambda_3 g(\Theta_1, \Theta_3)\xi + \frac{1}{n-1}S^G(\Theta_1, \Theta_3), \tag{34}$$

$$P^G(\Theta_1, \Theta_2)\xi = 0, \tag{35}$$

$$\eta\left(P^G(\Theta_1, \Theta_2)\Theta_3\right) = 0, \tag{36}$$

for all $\Theta_1, \Theta_2, \Theta_3 \in \chi(M)$.

Lemma 4. *The following relations hold for the concircular curvature tensor defined on an $n = (2m+1)$-dimensional Sasakian manifold admitting a general connection.*

$$C^G(\Theta_1,\Theta_2)\Theta_3 = R^G(\Theta_1,\Theta_2)\Theta_3 - \frac{r^G}{n(n-1)}\left[g(\Theta_2,\Theta_3)\Theta_1 - g(\Theta_1,\Theta_3)\Theta_2\right], \tag{37}$$

$$C^G(\xi,\Theta_2)\Theta_3 = A\left[-g(\Theta_2,\Theta_3)\xi + \eta(\Theta_3)\Theta_2\right], \tag{38}$$

$$C^G(\Theta_1,\xi)\Theta_3 = A\left[g(\Theta_1,\Theta_3)\xi - \eta(\Theta_3)\Theta_1\right], \tag{39}$$

$$C^G(\Theta_1,\Theta_2)\xi = A\left[\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1\right], \tag{40}$$

$$\eta\left(C^G(\Theta_1,\Theta_2)\Theta_3\right) = Ag\left(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2,\Theta_3\right), \tag{41}$$

for all $\Theta_1,\Theta_2,\Theta_3 \in \chi(M)$, where $A = \Lambda_3 + \frac{r^G}{n(n-1)}$.

3 Semi-Symmetric Sasakian Manifolds Admitting a General Connection

In this section, we first examine the conditions of projective flat and concircular flatness for Sasakian manifolds admitting a general connection and then obtain the characterizations of projective and concircular semi-symmetric manifolds.

Theorem 1. *Let M be an n-dimensional Sasakian manifold admitting a general connection. If M is a projective flat manifold, then M is an Einstein manifold.*

Proof. Suppose that M is a projective flat manifold. In this case, we have

$$P^G(\Theta_1,\Theta_2)\Theta_3 = 0,$$

for all $\Theta_1,\Theta_2,\Theta_3 \in \chi(M)$. If we use (32), we have

$$R^G(\Theta_1,\Theta_2)\Theta_3 = \frac{1}{n-1}\left[S^G(\Theta_2,\Theta_3)\Theta_1 - S^G(\Theta_1,\Theta_3)\Theta_2\right]. \tag{42}$$

If we choose $\Theta_2 = \xi$ in (42) and use $(18), (24)$, we obtain

$$S^G(\Theta_1, \Theta_3)\xi = -(n-1)\Lambda_3 g(\Theta_1, \Theta_3)\xi.$$

Taking the inner product of both sides of the last equality with the vector field $\xi \in \chi(M)$, we have

$$S^G(\Theta_1, \Theta_3) = -(n-1)\Lambda_3 g(\Theta_1, \Theta_3).$$

Thus, the proof is complete. □

Theorem 2. *Let M be an n-dimensional Sasakian manifold admitting a general connection. If M is a concircular flat manifold, then M is an real space form.*

Proof. The proof of the theorem is clear from the definition of the concircular curvature tensor. □

Theorem 3. *Let M be an n-dimensional Sasakian manifold admitting a general connection D^G. If M is a concircular semi-symmetric manifold, then M is either a flat or the general connection on M reduces to one of the Tanaka-Webster, Zamkovoy, or Schouten-van Kampen connections.*

Proof. Suppose that M is a concircular semi-symmetric manifold. In this case, we can write

$$\left(R^G(\Theta_1, \Theta_2) \cdot C^G\right)(\Theta_4, \Theta_5, \Theta_3) = 0,$$

for all $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 \in \chi(M)$. The meaning of this is

$$\begin{aligned} &R^G(\Theta_1, \Theta_2) C^G(\Theta_4, \Theta_5)\Theta_3 - C^G\left(R^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5\right)\Theta_3 \\ &-C^G\left(\Theta_4, R^G(\Theta_1, \Theta_2)\Theta_5\right)\Theta_3 - C^G(\Theta_4, \Theta_5) R^G(\Theta_1, \Theta_2)\Theta_3 = 0. \end{aligned} \tag{43}$$

If we choose $\Theta_1 = \xi$ in (43) and make use of (23), then we obtain

$$\begin{aligned} &-\Lambda_3 g\left(\Theta_2, C^G(\Theta_4, \Theta_5)\Theta_3\right)\xi + \Lambda_3 \eta\left(C^G(\Theta_4, \Theta_5)\Theta_3\right)\Theta_2 \\ &+\Lambda_3 g(\Theta_2, \Theta_4) C^G(\xi, \Theta_5)\Theta_3 - \Lambda_3 \eta(\Theta_4) C^G(\Theta_2, \Theta_5)\Theta_3 \\ &+\Lambda_3 g(\Theta_2, \Theta_5) C^G(\Theta_4, \xi)\Theta_3 - \Lambda_3 \eta(\Theta_5) C^G(\Theta_4, \Theta_2)\Theta_3 \\ &+\Lambda_3 g(\Theta_2, \Theta_3) C^G(\Theta_4, \Theta_5)\xi - \Lambda_3 \eta(\Theta_3) C^G(\Theta_4, \Theta_5)\Theta_2 = 0. \end{aligned} \tag{44}$$

If we use $(38), (39), (40)$ in (44), we have

$$\begin{aligned}
&-\Lambda_3 g\left(\Theta_2, C^G\left(\Theta_4, \Theta_5\right) \Theta_3\right) \xi+\Lambda_3 \eta\left(C^G\left(\Theta_4, \Theta_5\right) \Theta_3\right) \Theta_2 \\
&-A \Lambda_3 g\left(\Theta_2, \Theta_4\right) g\left(\Theta_5, \Theta_3\right) \xi+A \Lambda_3 g\left(\Theta_2, \Theta_4\right) \eta\left(\Theta_3\right) \Theta_5 \\
&-\Lambda_3 \eta\left(\Theta_4\right) C^G\left(\Theta_2, \Theta_5\right) \Theta_3+A \Lambda_3 g\left(\Theta_2, \Theta_5\right) g\left(\Theta_4, \Theta_3\right) \xi \\
&-A \Lambda_3 g\left(\Theta_2, \Theta_5\right) \eta\left(\Theta_3\right) \Theta_4-\Lambda_3 \eta\left(\Theta_5\right) C^G\left(\Theta_4, \Theta_2\right) \Theta_3 \\
&+A \Lambda_3 g\left(\Theta_2, \Theta_3\right) \eta\left(\Theta_4\right) \Theta_5-A \Lambda_3 g\left(\Theta_2, \Theta_3\right) \eta\left(\Theta_5\right) \Theta_4 \\
&-\Lambda_3 \eta\left(\Theta_3\right) C^G\left(\Theta_4, \Theta_5\right) \Theta_2=0.
\end{aligned} \tag{45}$$

If we choose $\Theta_4 = \xi$ in (45) , we obtain

$$-\Lambda_3\left[C^G\left(\Theta_2, \Theta_5\right) \Theta_3-A\left(g\left(\Theta_2, \Theta_3\right) \Theta_5-g\left(\Theta_5, \Theta_3\right) \Theta_2\right)\right]=0. \tag{46}$$

If we substitute (37) for (46) , we get

$$-\Lambda_3\left[R^G\left(\Theta_2, \Theta_5\right) \Theta_3-\Lambda_3\left(g\left(\Theta_2, \Theta_3\right) \Theta_5-g\left(\Theta_5, \Theta_3\right) \Theta_2\right)\right]=0.$$

This proves our assertion. □

Corollary 1. *Let M be an n-dimensional Sasakian manifold admitting a quarter-symmetric metric connection D^q. If M is a concircular semi-symmetric manifold, then M is a real space form.*

Theorem 4. *Let M be an n-dimensional Sasakian manifold admitting a general connection. If M is a projective semi-symmetric manifold, then M is a Einstein manifold provided $\Lambda_3 \neq 0$.*

Proof. Suppose that M is a projective semi-symmetric manifold. In this case, we can write

$$\left(R^G\left(\Theta_1, \Theta_2\right) \cdot P^G\right)\left(\Theta_4, \Theta_5, \Theta_3\right)=0,$$

for all $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 \in \chi(M)$. This implies that

$$\begin{aligned}
&R^G\left(\Theta_1, \Theta_2\right) P^G\left(\Theta_4, \Theta_5\right) \Theta_3-P^G\left(R^G\left(\Theta_1, \Theta_2\right) \Theta_4, \Theta_5\right) \Theta_3 \\
&-P^G\left(\Theta_4, R^G\left(\Theta_1, \Theta_2\right) \Theta_5\right) \Theta_3-P^G\left(\Theta_4, \Theta_5\right) R^G\left(\Theta_1, \Theta_2\right) \Theta_3=0.
\end{aligned} \tag{47}$$

If we choose $\Theta_1 = \xi$ in (47) and making use of (23), then we obtain

$$\begin{aligned} &-\Lambda_3 g\left(\Theta_2, P^G\left(\Theta_4, \Theta_5\right)\Theta_3\right)\xi + \Lambda_3\eta\left(P^G\left(\Theta_4, \Theta_5\right)\Theta_3\right)\Theta_2 \\ &+\Lambda_3 g\left(\Theta_2, \Theta_4\right) P^G\left(\xi, \Theta_5\right)\Theta_3 - \Lambda_3\eta\left(\Theta_4\right) P^G\left(\Theta_2, \Theta_5\right)\Theta_3 \\ &+\Lambda_3 g\left(\Theta_2, \Theta_5\right) P^G\left(\Theta_4, \xi\right)\Theta_3 - \Lambda_3\eta\left(\Theta_5\right) P^G\left(\Theta_4, \Theta_2\right)\Theta_3 \\ &+\Lambda_3 g\left(\Theta_2, \Theta_3\right) P^G\left(\Theta_4, \Theta_5\right)\xi - \Lambda_3\eta\left(\Theta_3\right) P^G\left(\Theta_4, \Theta_5\right)\Theta_2 = 0. \end{aligned} \tag{48}$$

By using $(33), (34), (35)$ in (48), we have

$$\begin{aligned} &-\Lambda_3 g\left(\Theta_2, P^G\left(\Theta_4, \Theta_5\right)\Theta_3\right)\xi + \Lambda_3\eta\left(P^G\left(\Theta_4, \Theta_5\right)\Theta_3\right)\Theta_2 \\ &-\Lambda_3^2 g\left(\Theta_2, \Theta_4\right) g\left(\Theta_5, \Theta_3\right)\xi - \tfrac{\Lambda_3}{n-1} g\left(\Theta_2, \Theta_4\right) S^G\left(\Theta_5, \Theta_3\right)\xi \\ &-\Lambda_3\eta\left(\Theta_4\right) P^G\left(\Theta_2, \Theta_5\right)\Theta_3 + \Lambda_3^2 g\left(\Theta_2, \Theta_5\right) g\left(\Theta_4, \Theta_3\right)\xi \\ &+\tfrac{\Lambda_3}{n-1} g\left(\Theta_2, \Theta_5\right) S^G\left(\Theta_4, \Theta_3\right)\xi - \Lambda_3\eta\left(\Theta_5\right) P^G\left(\Theta_4, \Theta_2\right)\Theta_3 \\ &-\Lambda_3\eta\left(\Theta_3\right) P^G\left(\Theta_4, \Theta_5\right)\Theta_2 = 0. \end{aligned} \tag{49}$$

If we choose $\Theta_4 = \xi$ in (49) and making use of (33), we get

$$\begin{aligned} &-\Lambda_3^2 g\left(\Theta_5, \Theta_3\right)\Theta_2 - \tfrac{\Lambda_3}{n-1} S^G\left(\Theta_5, \Theta_3\right)\Theta_2 - \Lambda_3 P^G\left(\Theta_2, \Theta_5\right)\Theta_3 \\ &+\Lambda_3^2 g\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_5\right)\xi + \tfrac{\Lambda_3}{n-1} S^G\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_5\right)\xi \\ &+\Lambda_3^2 g\left(\Theta_5, \Theta_2\right)\eta\left(\Theta_3\right)\xi + \tfrac{\Lambda_3}{n-1} S^G\left(\Theta_5, \Theta_2\right)\eta\left(\Theta_3\right)\xi = 0. \end{aligned} \tag{50}$$

If we choose $\Theta_3 = \xi$ in (50) and make use of $(18), (35)$, then we have

$$\Lambda_3^2 g\left(\Theta_5, \Theta_2\right)\xi + \frac{\Lambda_3}{n-1} S^G\left(\Theta_5, \Theta_2\right)\xi = 0.$$

If we take the inner product of both sides of the last equation with $\xi \in \chi(M)$, we get

$$S^G\left(\Theta_2, \Theta_5\right) = -\left(n-1\right)\Lambda_3 g\left(\Theta_2, \Theta_5\right).$$

This completes our proof. □

Corollary 2. *Let M be an n-dimensional Sasakian manifold admitting a quarter-symmetric metric connection D^q. If M is a projective semi-symmetric manifold, then M is a Einstein manifold.*

Theorem 5. *Let M be an n-dimensional Sasakian manifold admitting a general connection. If the curvature condition $C^G(\Theta_1,\Theta_2)\cdot P^G=0$ holds on the manifold M, then M is either an Einstein manifold provided $\Lambda_3\neq 0$ or a manifold with scalar curvature $r^G=-n(n-1)\Lambda_3$.*

Proof. Let us assume that

$$\left(C^G(\Theta_1,\Theta_2)\cdot P^G\right)(\Theta_4,\Theta_5,\Theta_3)=0$$

for all $\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5\in\chi(M)$. This gives us

$$\begin{aligned}&C^G(\Theta_1,\Theta_2)P^G(\Theta_4,\Theta_5)\Theta_3-P^G\left(C^G(\Theta_1,\Theta_2)\Theta_4,\Theta_5\right)\Theta_3\\&-P^G\left(\Theta_4,C^G(\Theta_1,\Theta_2)\Theta_5\right)\Theta_3-P^G(\Theta_4,\Theta_5)C^G(\Theta_1,\Theta_2)\Theta_3=0.\end{aligned}\tag{51}$$

If we choose $\Theta_1=\xi$ in (51), we have

$$\begin{aligned}&-Ag\left(\Theta_2,P^G(\Theta_4,\Theta_5)\Theta_3\right)\xi+A\eta\left(P^G(\Theta_4,\Theta_5)\Theta_3\right)\Theta_2\\&+Ag(\Theta_2,\Theta_4)P^G(\xi,\Theta_5)\Theta_3-A\eta(\Theta_4)P^G(\Theta_2,\Theta_5)\Theta_3\\&+Ag(\Theta_2,\Theta_5)P^G(\Theta_4,\xi)\Theta_3-A\eta(\Theta_5)P^G(\Theta_4,\Theta_2)\Theta_3\\&+Ag(\Theta_2,\Theta_3)P^G(\Theta_4,\Theta_5)\xi-A\eta(\Theta_3)P^G(\Theta_4,\Theta_5)\Theta_2=0.\end{aligned}\tag{52}$$

If we making use of (33), (34), (35) in (52), we get

$$\begin{aligned}&-Ag\left(\Theta_2,P^G(\Theta_4,\Theta_5)\Theta_3\right)\xi+A\eta\left(P^G(\Theta_4,\Theta_5)\Theta_3\right)\Theta_2\\&-A\Lambda_3g(\Theta_2,\Theta_4)g(\Theta_5,\Theta_3)\xi-\tfrac{A}{n-1}g(\Theta_2,\Theta_4)S^G(\Theta_5,\Theta_3)\xi\\&-A\eta(\Theta_4)P^G(\Theta_2,\Theta_5)\Theta_3+A\Lambda_3g(\Theta_2,\Theta_5)g(\Theta_4,\Theta_3)\xi\\&+\tfrac{A}{n-1}g(\Theta_2,\Theta_5)S^G(\Theta_4,\Theta_3)\xi-A\eta(\Theta_5)P^G(\Theta_4,\Theta_2)\Theta_3\\&-A\eta(\Theta_3)P^G(\Theta_4,\Theta_5)\Theta_2=0.\end{aligned}\tag{53}$$

If we choose $\Theta_4=\xi$ in (53) and using (33), we obtain

$$\begin{aligned}&-A\Lambda_3g(\Theta_5,\Theta_3)\Theta_2-\tfrac{A}{n-1}S^G(\Theta_5,\Theta_3)\Theta_2-AP^G(\Theta_2,\Theta_5)\Theta_3\\&+A\Lambda_3g(\Theta_2,\Theta_3)\eta(\Theta_5)\xi+\tfrac{A}{n-1}S^G(\Theta_2,\Theta_3)\eta(\Theta_5)\xi\\&+A\Lambda_3g(\Theta_5,\Theta_2)\eta(\Theta_3)\xi+\tfrac{A}{n-1}S^G(\Theta_5,\Theta_2)\eta(\Theta_3)\xi=0.\end{aligned}\tag{54}$$

Again, taking $\Theta_3 = \xi$ in (54) and use $(18), (35)$, we have

$$A\left[\Lambda_3 g(\Theta_5, \Theta_2)\xi + \frac{1}{n-1}S^G(\Theta_5, \Theta_2)\xi\right] = 0.$$

If we take the inner product of both sides of the last equation with $\xi \in \chi(M)$, we get

$$A\left[\Lambda_3 g(\Theta_5, \Theta_2) + \frac{1}{n-1}S^G(\Theta_5, \Theta_2)\right] = 0.$$

This completes our proof. □

Corollary 3. *Let M be an n-dimensional Sasakian manifold admitting a quarter-symmetric metric connection D^q. If the curvature condition $C^G(\Theta_1, \Theta_2) \cdot P^G = 0$ holds on the manifold M, then M is either an Einstein manifold or it has the scalar curvature $r^G = 2n(n-1)$.*

Theorem 6. *Let M be an n-dimensional Sasakian manifold admitting a general connection. If the curvature condition $P^G(\Theta_1, \Theta_2) \cdot C^G = 0$ holds on the manifold M, then M is either an Einstein manifold provided $\Lambda_3 \neq 0$ or a manifold with scalar curvature $r^G = -n(n-1)\Lambda_3$.*

Proof. Let us assume that

$$\left(P^G(\Theta_1, \Theta_2) \cdot C^G\right)(\Theta_4, \Theta_5, \Theta_3) = 0$$

for all $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 \in \chi(M)$. Thus, we can write

$$\begin{aligned} & P^G(\Theta_1, \Theta_2) C^G(\Theta_4, \Theta_5)\Theta_3 - C^G\left(P^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5\right)\Theta_3 \\ & -C^G\left(\Theta_4, P^G(\Theta_1, \Theta_2)\Theta_5\right)\Theta_3 - C^G(\Theta_4, \Theta_5) P^G(\Theta_1, \Theta_2)\Theta_3 = 0. \end{aligned} \tag{55}$$

If taking $\Theta_1 = \xi$ in (55), we have

$$\begin{aligned} & -\Lambda_3 g\left(\Theta_2, C^G(\Theta_4, \Theta_5)\Theta_3\right)\xi - \tfrac{1}{n-1}S^G\left(\Theta_2, C^G(\Theta_4, \Theta_5)\Theta_3\right)\xi \\ & +\Lambda_3 g(\Theta_2, \Theta_4) C^G(\xi, \Theta_5)\Theta_3 + \tfrac{1}{n-1}S^G(\Theta_2, \Theta_4) C^G(\xi, \Theta_5)\Theta_3 \\ & +\Lambda_3 g(\Theta_2, \Theta_5) C^G(\Theta_4, \xi)\Theta_3 + \tfrac{1}{n-1}S^G(\Theta_2, \Theta_5) C^G(\Theta_4, \xi)\Theta_3 \\ & +\Lambda_3 g(\Theta_2, \Theta_3) C^G(\Theta_4, \Theta_5)\xi + \tfrac{1}{n-1}S^G(\Theta_2, \Theta_3) C^G(\Theta_4, \Theta_5)\xi = 0 \end{aligned} \tag{56}$$

Making use of $(38), (39), (40)$ in (56), we get

$$\begin{aligned}
&-\Lambda_3 g\left(\Theta_2, C^G\left(\Theta_4, \Theta_5\right)\Theta_3\right)\xi - \tfrac{1}{n-1}S^G\left(\Theta_2, C^G\left(\Theta_4, \Theta_5\right)\Theta_3\right)\xi \\
&-A\Lambda_3 g\left(\Theta_2, \Theta_4\right) g\left(\Theta_5, \Theta_3\right)\xi + A\Lambda_3 g\left(\Theta_2, \Theta_4\right)\eta\left(\Theta_3\right)\Theta_5 \\
&-\tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_4\right) g\left(\Theta_5, \Theta_3\right)\xi + \tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_4\right)\eta\left(\Theta_3\right)\Theta_5 \\
&+A\Lambda_3 g\left(\Theta_2, \Theta_5\right) g\left(\Theta_4, \Theta_3\right)\xi - A\Lambda_3 g\left(\Theta_2, \Theta_5\right)\eta\left(\Theta_3\right)\Theta_4 \\
&+\tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_5\right) g\left(\Theta_4, \Theta_3\right)\xi - \tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_5\right)\eta\left(\Theta_3\right)\Theta_4 \\
&+A\Lambda_3 g\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_4\right)\Theta_5 - A\Lambda_3 g\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_5\right)\Theta_4 \\
&+\tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_4\right)\Theta_5 - \tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_5\right)\Theta_4 = 0.
\end{aligned} \tag{57}$$

If we choose $\Theta_4 = \xi$ in (57) and by using $(18), (38)$, we obtain

$$\begin{aligned}
&-\tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_5\right)\eta\left(\Theta_3\right)\xi - A\Lambda_3 g\left(\Theta_2, \Theta_5\right)\eta\left(\Theta_3\right)\xi \\
&+A\Lambda_3 g\left(\Theta_2, \Theta_3\right)\Theta_5 - A\Lambda_3 g\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_5\right)\xi \\
&+\tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_3\right)\Theta_5 - \tfrac{A}{n-1}S^G\left(\Theta_2, \Theta_3\right)\eta\left(\Theta_5\right)\xi = 0.
\end{aligned} \tag{58}$$

If we choose $\Theta_3 = \xi$ in (58) and by using (18), we have

$$A\left[\Lambda_3 g\left(\Theta_5, \Theta_2\right)\xi + \frac{1}{n-1}S^G\left(\Theta_5, \Theta_2\right)\xi\right] = 0.$$

This completes our proof. □

Corollary 4. *Let M be an n-dimensional Sasakian manifold admitting a quarter-symmetric metric connection D^q. If the curvature condition $P^G\left(\Theta_1, \Theta_2\right)\cdot C^G = 0$ holds on the manifold M, then M is either an Einstein manifold or a manifold with scalar curvature $r^G = 2n\left(n-1\right)$.*

4 η-Ricci-Bourguignon Solitons on Sasakian Manifolds Admitting General Connection

Ricci-Bourguignon solitons are related to geometric flow theory, which is important in fields such as differential geometry and general relativity. They can be considered a generalization of Ricci solitons, which study the evolution of a manifold under Ricci flow. These solitons, modified by the Bourguignon tensor, are used to understand different classes of Riemannian geometry.

A Ricci-Bourguignon soliton on a semi-Riemannian manifold (M, g) is a data (g, Θ_5, λ) fulfilling

$$L_V g + 2S + (2\lambda - \rho r)\, g = 0, \tag{59}$$

where L denote the Lie-derivative, S and r denote Ricci tensor and scalar curvature, respectively, and λ is real constants, ρ is a nonzero constant.

An η-Ricci Bourguignon soliton on (M, g) is a data (g, V, λ, μ) fulfilling

$$L_V g + 2S + (2\lambda - \rho r)\, g + 2\mu\eta \otimes \eta = 0, \tag{60}$$

where L denote the Lie-derivative, S and r are Ricci tensor and scalar curvatures, respectively, and λ, μ are real constants, ρ is a nonzero constant. The η-Ricci Bourguignon soliton is said to be expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$.

These solitons are used to determine the stationary points of geometric flows and to study the characterization of manifolds with various curvature conditions. They are used in studying the geometric structures of space-time in relation to Einstein's field equations. They are used in studying the geometric structures of space-time in relation to Einstein's field equations.

We consider a Sasakian manifold admitting general connection admitting an η-Ricci Bourguignon soliton (g, ξ, λ, μ). Then from (60), it obvious that

$$\left(L_\xi^G g\right)(\Theta_1, \Theta_2) + 2S^G(\Theta_1, \Theta_2) + \left(2\lambda - \rho r^G\right) g(\Theta_1, \Theta_2) + 2\mu\eta(\Theta_1)\,\eta(\Theta_2) = 0. \tag{61}$$

Next, we will express the Lie derivative along ξ on M admitting general connection as follows:

$$\begin{aligned}\left(L_\xi^G g\right)(\Theta_1, \Theta_2) &= L_\xi^G g(\Theta_1, \Theta_2) - g\left(L_\xi^G \Theta_1, \Theta_2\right) - g\left(\Theta_1, L_\xi^G \Theta_2\right) \\ &= L_\xi^G g(\Theta_1, \Theta_2) - g\left([\xi, \Theta_1]_G, \Theta_2\right) - g\left(\Theta_1, [\xi, \Theta_2]_G\right).\end{aligned}$$

By means of (4) and (15) , the last equation reduces to

$$\left(L_\xi^G g\right)(\Theta_1, \Theta_2) = 0. \tag{62}$$

By virtue of (62), the equation (61) takes the following form

$$S^G(\Theta_1, \Theta_2) = \left(\rho r^G - \lambda\right) g(\Theta_1, \Theta_2) - \mu\eta(\Theta_1)\,\eta(\Theta_2). \tag{63}$$

Thus, we can state the following theorem.

Theorem 7. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting a general connection and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. Then M is an η-Einstein manifold provided $\lambda \neq \rho r^G$ and $\mu \neq 0$.*

Particularly, if $\lambda \neq \rho r^G$ and $\mu = 0$, Sasakian manifold admitting general connection M admitting η-Ricci Bourguignon soliton reduces to Einstein manifold.

If we choose $\Theta_2 = \xi$ in (63), then we have

$$S^G(\Theta_1, \xi) = \left[\rho r^G - (\lambda + \mu)\right] \eta(\Theta_1),$$

and we can state the following corollary.

Corollary 5. *Let M be a Sasakian manifold admitting almost η-Ricci Bourguignon soliton admitting general connection D^G, then λ and μ are related by*

$$\lambda + \mu = \rho r^G + (n-1)\Lambda_3. \tag{64}$$

Using (64), we will characterize λ and μ in the Sasakian manifold admitting the almost η-Ricci Bourguignon soliton according to different connections as follows.

Theorem 8. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold and (g, ξ, λ, μ) be an almost η-Ricci Bourguignon soliton on M. Then the following holds:*
i. For quarter-symmetric metric connection D^q

$$\lambda + \mu = \rho r^G - 2(n-1),$$

ii. For Schouten-Van Kampen connection D^S

$$\lambda + \mu = \rho r^G,$$

iii. For Zamkovoy connection D^Z

$$\lambda + \mu = \rho r^G,$$

iv. For generalized Tanaka Webster connection D^T

$$\lambda + \mu = \rho r^G.$$

Definition 1. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection D^G. If $R^G \cdot S^G$ and $Q^G(g, S^G)$ are linearly dependent, then the M is said to be* ***Ricci pseudosymmetric****.*

In this case, there exists a function L_R on M such that

$$R^G \cdot S^G = \mathcal{L}_R Q^G \left(g, S^G\right).$$

In particular, if $\mathcal{L}_R = 0$, the M is said to be **Ricci semisymmetric**.

Theorem 9. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection D^g and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a Ricci pseudosymmetric, then at least one of the following holds:*
i. $\mathcal{L}_R = -\Lambda_3$,
ii. $\lambda = (n-1)\Lambda_3 + \rho r^G$ *and* $\mu = 0$,
iii. M is an expanding if $\rho r^G > (1-n)\Lambda_3$,
iv. M is a steady if $\rho r^G = (1-n)\Lambda_3$,
v. M is a shrinking if $\rho r^G < (1-n)\Lambda_3$,
vi. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to a Ricci-Bourguignon soliton (g, ξ, λ).

Proof. Let's assume that M is a Ricci pseudosymmetric and (g, ξ, λ, μ) be almost η-Ricci Bourguignon soliton on M admitting general connection. That's mean

$$\left(R^G(\Theta_1, \Theta_2) \cdot S^G\right)(\Theta_4, \Theta_5) = \mathcal{L}_R Q^G \left(g, S^G\right)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(TM)$, which is equivalent to

$$\begin{aligned} & S^G\left(R^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5\right) + S^G\left(\Theta_4, R^G(\Theta_1, \Theta_2)\Theta_5\right) \\ & = \mathcal{L}_R\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\right\}. \end{aligned} \tag{65}$$

If we choose $\Theta_5 = \xi$ in (65), we get

$$\begin{aligned} & S^G\left(R^G(\Theta_1, \Theta_2)\Theta_4, \xi\right) + S^G\left(\Theta_4, R^G(\Theta_1, \Theta_2)\xi\right) \\ & = \mathcal{L}_R\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \xi) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\xi)\right\}. \end{aligned}$$

Making (18) and (22) in the last equality, one can easily to see

$$\begin{aligned} & -(n-1)\Lambda_3^2 g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4) \\ & +\Lambda_3 S^G(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & = \mathcal{L}_R\{-(n-1)\Lambda_3 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & + S^G(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\}. \end{aligned} \tag{66}$$

Putting (63) in (66), we get

$$\left[(n-1)\Lambda_3 + \left(\rho r^G - \lambda\right)\right]\left[\Lambda_3 + \mathcal{L}_R\right] g\left(\eta\left(\Theta_1\right)\Theta_2 - \eta\left(\Theta_2\right)\Theta_1, \Theta_4\right) = 0.$$

This completes the proof. □

We can give some important results of this theorem as follows.

Corollary 6. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting general connection D^G and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a Ricci semisymmetric, then at least one of the following holds:*
i. $\lambda = (n-1)\Lambda_3 + \rho r^G$ *and* $\mu = 0$,
ii. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to a Ricci-Bourguignon soliton (g, ξ, λ),
iii. The general connection D^G on M reduces to one of the Tanaka-Webster D^T, Zamkovoy D^Z, or Schouten-van Kampen D^S connections,
iv. M is an expanding if $\rho r^G > (1-n)\Lambda_3$,
v. M is a steady if $\rho r^G = (1-n)\Lambda_3$,
vi. M is a shrinking if $\rho r^G < (1-n)\Lambda_3$.

Corollary 7. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting quarter-symmetric metric connection D^q and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a Ricci pseudosymmetric, then at least one of the following holds:*
i. $\mathcal{L}_R = 2$,
ii. $\lambda = \rho r^G - 2(n-1)$ *and* $\mu = 0$,
iii. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to a Ricci-Bourguignon soliton (g, ξ, λ),
iv. M is an expanding if $\rho r^G > 2(n-1)$,
v. M is a steady if $\rho r^G = 2(n-1)$,
vi. M is a shrinking if $\rho r^G < 2(n-1)$.

Corollary 8. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M admitting by any of the connections generalized Tanaka Webster, Zamkovoy or Schouten-Van Kampen. If M is a Ricci pseudosymmetric, then at least one of the following holds:*
i. M is a Ricci semisymmetric,
ii. $\lambda = \rho r^G$ *and* $\mu = 0$,
iii. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to a Ricci-Bourguignon soliton (g, ξ, λ),
iv. M is an expanding if $\rho r^G > 0$,
v. M is a steady if $\rho r^G = 0$,
vi. M is a shrinking if $\rho r^G < 0$.

Definition 2. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting general connection D^G. If $P^G \cdot S^G$ and $Q^G\left(g, S^G\right)$ are linearly dependent, then M is said to be* ***projective Ricci pseudosymmetric.***

In this case, there exists a function $\mathcal{L}_{PG}$ on M such that

$$P^G \cdot S^G = \mathcal{L}_{PG} Q^G\left(g, S^G\right).$$

In particular, if $\mathcal{L}_{PG} = 0$, the M is said to be **projective Ricci semisymmetric.**

Now, let us investigate the condition for an n-dimensional Sasakian manifold admitting a general connection D^G to be projective flat.

Theorem 10. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting general connection D^G and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a projective flat, then at least one of the following holds:*
i. $\lambda = (n-1)\Lambda_3 + \rho r^G$ *and* $\mu = 0$,
ii. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to a Ricci-Bourguignon soliton (g, ξ, λ),
iii. M is an Einstein manifold,
iv. M is an expanding if $\rho r^G > (1-n)\Lambda_3$,
v. M is a steady if $\rho r^G = (1-n)\Lambda_3$,
vi. M is a shrinking if $\rho r^G < (1-n)\Lambda_3$.

Proof. The proof of the theorem is clear from Theorem 1 and (63). □

Corollary 9. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting quarter-symmetric metric connection D^q and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a projective flat, then at least one of the following holds:*
i. M is an Einstein manifold,
ii. $\lambda = \rho r^G - 2(n-1)$ *and* $\mu = 0$,
iii. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to a Ricci-Bourguignon soliton (g, ξ, λ),
iv. M is an expanding if $\rho r^G > 2(n-1)$,
v. M is a steady if $\rho r^G = 2(n-1)$,
vi. M is a shrinking if $\rho r^G < 2(n-1)$.

Theorem 11. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting general connection D^G and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a projective Ricci pseudosymmetric, then at least one of the following holds:*
i) M is a projective Ricci semi symmetric.
ii) $\lambda = (n-1)\Lambda_3 + \rho r^G$ *and* $\mu = 0$,

iii) The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to Ricci-Bourguignon soliton (g, ξ, λ).
iii) M is an expanding if $\rho r^G > (1-n)\Lambda_3$,
iv) M is a steady if $\rho r^G = (1-n)\Lambda_3$,
v) M is a shrinking $\rho r^G < (1-n)\Lambda_3$.

Proof. Let's assume that $n = (2m+1)$-dimensional Sasakian manifold M be a projective Ricci pseudosymmetric and (g, ξ, λ, μ) be almost η-Ricci Bourguignon soliton on M admitting general connection. That's mean

$$\left(P^G(\Theta_1, \Theta_2) \cdot S^G\right)(\Theta_4, \Theta_5) = \mathcal{L}_{P^G} Q^G\left(g, S^G\right)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} & S^G\left(P^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5\right) + S^G\left(\Theta_4, P^G(\Theta_1, \Theta_2)\Theta_5\right) \\ & = \mathcal{L}_{P^G}\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\right\}. \end{aligned} \tag{67}$$

If we choose $\Theta_5 = \xi$ in (67), we get

$$\begin{aligned} & S^G\left(P^G(\Theta_1, \Theta_2)\Theta_4, \xi\right) + S^G\left(\Theta_4, P^G(\Theta_1, \Theta_2)\xi\right) \\ & = \mathcal{L}_{P^G}\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \xi) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\xi)\right\}. \end{aligned}$$

By using $(18), (35)$ and (36) in the last equality, we have

$$\begin{aligned} & \mathcal{L}_{P^G}\{-(n-1)\Lambda_3 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & + S^G(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\} = 0. \end{aligned} \tag{68}$$

If we use (63) in (68), we have

$$\left[(n-1)\Lambda_3 + \left(\rho r^G - \lambda\right)\right] \mathcal{L}_{P^G} g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) = 0.$$

This completes the proof. □

We can give the following results as follows.

Corollary 10. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting quarter-symmetric metric connection D^q and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a projective Ricci pseudosymmetric, then at least one of the following holds:*
i. M is a projective Ricci semisymmetric,
ii. $\lambda = \rho r^G - 2(n-1)$ and $\mu = 0$,

iii. The η-Ricci Bourguignon soliton (g,ξ,λ,μ) reduces to Ricci-Bourguignon soliton (g,ξ,λ).
iv. M is an expanding if $\rho r^{G} > 2(n-1)$,
v. M is a steady if $\rho r^{G} = 2(n-1)$,
vi. M is a shrinking if $\rho r^{G} < 2(n-1)$.

Corollary 11. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold and (g,ξ,λ,μ) be an η-Ricci Bourguignon soliton on M admitting by any of the connections generalized Tanaka Webster, Zamkovoy or Schouten-van Kampen. If M is a projective Ricci pseudosymmetric, then at least one of the following holds:*
i. M is a projective Ricci semisymmetric,
ii. η-Ricci Bourguignon soliton (g,ξ,λ,μ) reduces to Ricci-Bourguignon soliton (g,ξ,λ),
iii. $\lambda = \rho r^{G}$ and $\mu = 0$,
iv. M is an expanding if $\rho r^{G} > 0$,
v. M is a steady if $\rho r^{G} = 0$,
vi. M is a shrinking if $\rho r^{G} < 0$.

Definition 3. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold admitting general connection D^{G}. If $C^{G}\cdot S^{G}$ and $Q^{G}\left(g,S^{G}\right)$ are linearly dependent, then M is said to be* ***concircular Ricci pseudosymmetric.***

In this case, there exists a function such as $\mathcal{L}_{C^{G}}$ on M such that

$$C^{G}\cdot S^{G} = \mathcal{L}_{C^{G}} Q^{G}\left(g,S^{G}\right).$$

In particular, if $\mathcal{L}_{C^{G}} = 0$, the M is said to be **concircular Ricci semisymmetric.**

Theorem 12. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold admitting general connection D^{G} and (g,ξ,λ,μ) be an η-Ricci Bourguignon soliton on M. If M is a concircular Ricci pseudosymmetric, then at least one of the following holds:*
i. $\lambda = (n-1)\Lambda_3 + \rho r^{G}$ and $\mu = 0$,
ii. The η-Ricci Bourguignon soliton (g,ξ,λ,μ) reduces to Ricci-Bourguignon soliton (g,ξ,λ).
iii. $\mathcal{L}_{C^{G}} = -\left[\Lambda_3 + \frac{r^{G}}{n(n-1)}\right]$,
iv. M is an expanding if $\rho r^{G} > (1-n)\Lambda_3$,
v. M is a steady if $\rho r^{G} = (1-n)\Lambda_3$,
vi. M is a shrinking $\rho r^{G} < (1-n)\Lambda_3$.

Proof. Let's assume that $n=(2m+1)$-dimensional Sasakian manifold M be a concircular Ricci pseudosymmetric and (g,ξ,λ,μ) be almost η-Ricci Bourguignon soliton on M admitting general connection. That's mean

$$\left(C^{G}(\Theta_1,\Theta_2)\cdot S^{G}\right)(\Theta_4,\Theta_5) = \mathcal{L}_{C^{G}} Q^{G}\left(g,S^{G}\right)(\Theta_4,\Theta_5;\Theta_1,\Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(TM)$. This calculation give us

$$\begin{aligned} & S^G\left(C^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5\right) + S^G\left(\Theta_4, C^G(\Theta_1, \Theta_2)\Theta_5\right) \\ & = \mathcal{L}_{C^G}\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\right\}. \end{aligned} \tag{69}$$

If taking $\Theta_5 = \xi$ in (69), we get

$$\begin{aligned} & S^G\left(C^G(\Theta_1, \Theta_2)\Theta_4, \xi\right) + S^G\left(\Theta_4, C^G(\Theta_1, \Theta_2)\xi\right) \\ & = \mathcal{L}_{C^G}\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \xi) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\xi)\right\}. \end{aligned}$$

If making use of $(18), (40)$ and (41) in the last equality, we have

$$\begin{aligned} & -(n-1)A\Lambda_3 g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4) \\ & + AS^G(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & = \mathcal{L}_{C^G}\{-(n-1)\Lambda_3 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & + S^G(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\right\}. \end{aligned} \tag{70}$$

If we use (63) in (70), we have

$$\left[(n-1)\Lambda_3 + \left(\rho r^G - \lambda\right)\right][A + \mathcal{L}_{C^G}]\, g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) = 0.$$

This proves our assertions. □

We can give some important results of this theorem as follows.

Corollary 12. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection D^G and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a concircular Ricci semisymmetric, then at least one of the following holds:*
i. M is of scalar curvature $r^G = -n(n-1)\Lambda_3$,
ii. $\lambda = (n-1)\Lambda_3 + \rho r^G$ and $\mu = 0$,
iii. The η-Ricci Bourguignon soliton (g, ξ, λ, μ) reduces to Ricci-Bourguignon soliton (g, ξ, λ),
iv. M is an expanding if $\rho r^G > (1-n)\Lambda_3$,
v. M is a steady if $\rho r^G = (1-n)\Lambda_3$,
vi. M is a shrinking if $\rho r^G < (1-n)\Lambda_3$.

Corollary 13. *Let M be an $n = (2m+1)$-dimensional Sasakian manifold admitting quarter-symmetric metric connection D^q and (g, ξ, λ, μ) be an η-Ricci Bourguignon soliton on M. If M is a concircular Ricci*

pseudosymmetric, then at least one of the following holds:
i. $\mathcal{L}_{C^G} = -\left[\frac{r^G}{n(n-1)} - 2\right]$,
ii. $\lambda = \rho r^G - 2(n-1)$ *and* $\mu = 0$,
iii. The η*-Ricci Bourguignon soliton* (g, ξ, λ, μ) *reduces to Ricci-Bourguignon soliton* (g, ξ, λ),
iv. M *is an expanding if* $\rho r^G > 2(n-1)$,
v. M *is a steady if* $\rho r^G = 2(n-1)$,
vi. M *is a shrinking if* $\rho r^G < 2(n-1)$.

Corollary 14. *Let* M *be an* $n = (2m+1)$*-dimensional Sasakian manifold and* (g, ξ, λ, μ) *be an* η*-Ricci Bourguignon soliton on* M *admitting by any of the connections generalized Tanaka Webster, Zamkovoy or Schouten-Van Kampen. If* M *is a concircular Ricci pseudosymmetric, then at least one of the following holds:*
i. $L_{C^G} = -\frac{r^G}{n(n-1)}$,
ii. $\lambda = \rho r^G$ *and* $\mu = 0$,
iii. The η*-Ricci Bourguignon soliton* (g, ξ, λ, μ) *reduces to Ricci-Bourguignon soliton* (g, ξ, λ),
iii. M *is an expanding if* $\rho r^G > 0$,
iv. M *is a steady if* $\rho r^G = 0$,
v. M *is a shrinking if* $\rho r^G < 0$.

Theorem 13. *Let* M *be an* $n = (2m+1) -$*dimensional Sasakian manifold admitting general connection* D^G *and* (g, ξ, λ, μ) *be an* η*-Ricci Bourguignon soliton on* M*. If* M *is a concircular flat, then* M *is of scalar curvature* $r^G = -n(n-1)\Lambda_3$.

Proof. The proof of the theorem is a direct calculation. □

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