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The Number of Conjugacy Classes and Irreducible Characters in a Finite Group

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Abstract

In this paper, number of conjugacy classes and irreducible characters in a non-abelian group of order 2^6 are investigated using cycle pattern of elements. Through the exploits of commutator and representation of elements as a product of disjoint cycles, the number of conjugacy classes is obtained which extends some results in literature.

1. Introduction

The concept of conjugacy classes and irreducible characters which started since the time of Burnside (1852-1927) and Frobenius (1849-1917), at the present time plays a more important role in the study of groups and their representations. In recent time, the exploration of conjugacy classes has been the active subject and remains a major area of

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attention for many researchers as observed in [1] and [2]. The important part of this group under discussion is the conjugacy classes of a non-abelian group of order sixty four (2^6).

Let G be a finite group. Two elements a and b of G are conjugate if there exists $g \in G$ such that $gag^{-1} = b$. Conjugacy classes of finite group S_n can be represented using a cycle type which also shows the partition of n as demonstrated in [3]. [4], [5] and [6] had worked on conjugacy classes of finite group with resounding results. Motivated by the above literature and on-going research in this direction, the objective of this paper is to find the number of conjugacy classes and irreducible characters in a finite group of non-abelian group of order 2^6. Definitions and theorems corresponding to conjugacy classes are presented with proofs. Furthermore, conjugacy classes of a finite group together with their cycle pattern are presented using three subgroups of G, where G is non-abelian.

2. Preliminaries

Definition 2.1. Let G be a finite group. Then a conjugacy class $C(G)$ is a non-empty subset of G such that the following holds:

(i) Given any $x, y \in C$ there exists $g \in G$ such that $gxg^{-1} = y$.

(ii) If $x \in C$ and $g \in G$, then $gxg^{-1} = C$. In other words, it is closed under the action of group on itself.

Definition 2.2. Let G be a finite group consisting of n elements. Then a permutation group of degree n is a one to one mapping of S_n onto itself.

Definition 2.3. Let G be a finite group. Then the group of all permutations of G is the symmetric group on n letter which is denoted by S_n.

Definition 2.4. An irreducible character is a representation that has no non-trivial invariant subspace.

Proposition 2.1. *Any permutation of a finite set containing at least two elements can be written as the product of transpositions.*

Theorem 2.1. *Every permutation in* S_n *can be written as the product of disjoint cycles.*

Proof. We can assume that $X = \{1, 2, ..., n\}$. Let $\sigma \in S_n$ and $X_1 = \{\sigma(1), \sigma^2(1), ...\}$. The set X_1 is finite since X is finite. Now let i be the first integer in X that is not in X_1 and $X_2 = \{\sigma(i), \sigma^2(i), ...\}$. Again X_2 is a finite set. Continuing in that manner, we can define finite disjoint sets $X_3, X_4, \ldots$. Since X is finite set, we are guaranteed that this process will end and there will be only a finite number of these sets say γ. If σ_i is the cycle defined by

$$\sigma_i(x) = \begin{cases} \sigma(x) & x \in X_i, \\ x & x \notin X_i, \end{cases} \qquad \text{for } i = 1, 2, ..., r \tag{1.1}$$

then $\sigma = \sigma_1, \sigma_2, ..., \sigma_r$. Since the set $X_1, X_2, ..., X_r$ are disjoint, the cycles $\sigma_1, \sigma_2, ..., \sigma_r$ must be disjoint. See [7].

Theorem 2.2. *Let* (G, o) *be a group. Then, a conjugacy relation of G is an equivalence relation on G.*

Proof. Let $(\sim)$ be a conjugacy relation defined on G. Then it is required to show that $(\sim)$ is reflexive, symmetry and transitive. $\forall x \in G$ there exists e_G such that

$$e_G o x = x o e_G \Rightarrow x \sim x, \tag{1.2}$$

where e_G is the identity in G. Thus conjugacy relation of G is reflexive. $\forall x, y \in G$ there exists $a \in G$ such that

$$x \sim y \Rightarrow a o x = y o a. \tag{1.3}$$

Post multiply (1.3) by a^{-1} yields

$$a o x o a^{-1} = y \tag{1.4}$$

which implies that $y \sim x$. Thus conjugacy relation of G is symmetric. $\forall x, y, z \in G$ there exist a_1 and a_2 such that

$$x \sim y \Rightarrow a_1 o x = y o a_1. \tag{1.5}$$

$$y \sim z \Rightarrow a_2 o y = z o a_2. \tag{1.6}$$

From (1.5) we have

$$a_1 o x o a_1^{-1} = y. \tag{1.7}$$

Putting (1.7) in (1.6) gives

$$a_2 o a_1 o x o a_1^{-1} = z o a_2. \tag{1.8}$$

Post multiply (1.8) by a_1 gives

$$a_2 o a_1 o x o a_1^{-1} o a_1 = z o a_2 o a_1, \tag{1.9}$$

$$a_2 o a_1 o x e = z o a_2 o a_1, \tag{1.10}$$

$$a_2 o a_1 o x = z o a_2 o a_1 \tag{1.11}$$

which implies that $x \sim z$. Thus conjugacy relation of G is transitive. Hence conjugacy relation is an equivalence relation.

Remark. Since conjugacy relation is an equivalence relation, it partitions elements of the group into equivalence classes. This means that every element of the group belongs precisely to one conjugacy class and classes $C|(x)$ and $C|(y)$ are equal if and only if x and y are conjugate and disjoint otherwise. The equivalence class that contains $x \in G$ is $C|(x) = \{gxg^{-1} : g \in G\}$ and is called the *conjugacy class* of x.

Definition 2.5. The commutator of two elements w and p of a group G is the element $[w, p] = w^{-1}p^{-1}wp$ and it is equal to group identity if and only if w and p commute. The set of all commutators of a group is not in general closed under the group operation but the subgroup of G generated by all commutators is closed and is called the *commutator subgroup* of G.

3. Results

3.1. Conjugacy classes of *G* of order 2^6

We consider a non-abelian group $G = \{x, y\}$ with three subgroups p, q, r of G defined as $p = (12)$, $q = (13)(24)$ and $r = (15)(26)(37)(48)$. Let $x = pq$ and $y = qr$ such that $pq \neq qr$. Representing G as a product of disjoint cycles using some commutator identities and element of G: x, y, $[x, y]$, $[x, y]^2$, xy, x^2, y^{-1}, $x[x, y]^2$,

$(xy)^2$, xy^2, y^2 and xy^{-1}, we have

(i) $x = pq = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix} = (1432)$

(ii) $y = qr = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} = (1735)(2846)$

(iii) $x^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 5 & 6 & 7 & 8 \end{pmatrix} = (12)(34)$

(iv) $y^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix} = (13)(24)(57)(68)$

(v) $xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix} = (1625)(37)(48)$

(vi) $(xy)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 6 & 5 & 7 & 8 \end{pmatrix} = (12)(56)$

(vii) $xy^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 7 & 8 & 5 & 6 \end{pmatrix} = (12)(57)(68)$

(viii) $xy^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix} = (1827)(35)(46)$

(ix) $y^{-1} = y^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 3 & 4 & 1 & 2 \end{pmatrix} = (1537)(2648)$

(x) $x^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 5 & 6 & 7 & 8 \end{pmatrix} = (1324)$

(xi) $[x, y] = x^{-1}y^{-1}xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 7 & 8 & 6 & 5 \end{pmatrix} = (1324)(5768)$

(xii) $[x, y]^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix} = (12)(34)(56)(78)$

(xiii) $x[x, y]^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix} = (1324)(56)(78).$

Using (i) to (ii) the conjugacy classes of G with their cycle patterns are;

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix},$$

$$x^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 5 & 6 & 7 & 8 \end{pmatrix},$$

$$y^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix},$$

$$xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix},$$

$$(xy)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 6 & 5 & 7 & 8 \end{pmatrix}$$

$$xy^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 7 & 8 & 5 & 6 \end{pmatrix},$$

$$xy^3 = xy^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix},$$

$$[x, y] = x^{-1}y^{-1}xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 7 & 8 & 6 & 5 \end{pmatrix},$$

$$[x, y]^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix},$$

$$x[x, y]^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix},$$

$$y^{-1} = y^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

3.2. Element in each conjugacy class of group of order 2^6

Let C denote the classes. Then the elements in each class are outlined as follows;

$C1$: e where e is the identity element which always an element in every group.

$C2: Cl[x, y]^2$

$[x, y]^2 = (12)(34)(56)(78)$

$C3: Cl(x^2)$

$x^2 = (12)(34), \quad (x^2)^y = (56)(78)$

$C4: Cl([x, y])$

$[x, y] = (1324)(5768), \quad [x, y]^y = (1324)(5867),$

$[x, y]^{y^2} = (1423)(5867), \quad [x, y]^{y^3} = (1423)(5768)$

$C5: Cl(x)$

$x = (1423), \quad x^y = (5768), \quad x^{y^2} = (1324), \quad x^{y^3} = (5867)$

$C6: Cl(x[x, y]^2)$

$x[x, y]^2 = (1324)(56)(78), \; (x[x, y]^2)^y = (5867)(12)(34),$

$(x[x, y]^2)^{y^2} = (1423)(56)(78), \quad (x[x, y]^2)^{y^3} = (5768)(12)(34)$

$C7: Cl(y^2)$

$y^2 = (13)(24)(57)(68), \quad (y^2)^x = (13)(24)(57)(68),$

$(y^2)^{xy} = (13)(24)(58)(67), \quad (y^2)^{x^2y} = (14)(23)(58)$

$C8: Cl((xy)^2)$

$(xy)^2 = (12)(56), \quad ((xy)^2)^x = (34)(56),$

$((xy)^2)^y = (12)(78), \quad ((xy)^2)^{y^2} = (34)(78)$

$C9$: $Cl(xy^2)$

$xy^2 = (12)(57)(68)$, $(xy^2)^x = (34)(57)(68)$,

$(xy^2)^y = (13)(24)(78)$, $(xy^2)^{xy} = (13)(24)(56)$,

$(xy^2)^{yx} = (14)(23)(78)$, $(xy^2)^{y^2x} = (34)(58)(67)$,

$(xy^2)^{y^2x^2} = (12)(58)(67)$, $(xy^2)^{y^3x} = (14)(23)(56)$

$C10$: $Cl(y)$

$y = (1735)(2846)$, $y^x = (1547)(2638)$,

$y^{x^2} = (1836)(2745)$, $y^{x^3} = (1648)(2537)$,

$y^{xy} = (1637)(2548)$, $y^{x^2y} = (1746)(2835)$,

$y^{x^2y^2} = (1845)(2736)$, $y^{x^3y} = (1538)(2647)$

$C11$: $Cl(y^{-1})$

$y^{-1} = (1537)(2648)$, $(y^{-1})^x = (1745)(2836)$,

$(y^{-1})^{x^2} = (1638)(2547)$, $(y^{-1})^{x^3} = (1846)(2735)$,

$(y^{-1})^{xy} = (1736)(2845)$, $(y^{-1})^{x^3y} = (1835)(2746)$,

$(y^{-1})^{x^2y} = (1647)(2538)$, $(y^{-1})^{x^2y^2} = (1548)(2637)$

$C12$: $Cl(xy)$

$xy = (1625)(37)(48)$, $(xy)^x = (3546)(17)(48)$,

$(xy)^{x^2} = (1526)(38)(47)$, $(xy)^{x^3} = (3645)(18)(27)$,

$(xy)^y = (17278)(35)(46)$, $(xy)^{y^2} = (3847)(15)(26)$,

$(xy)^{x^2 y} = (1827)(36)(45), \quad (xy)^{yx^3} = (3748)(16)(25)$

$C13$: $Cl(xy^{-1})$

$xy^{-1} = (1827)(35)(46), \quad (xy^{-1})^{x} = (3748)(15)(26),$

$(xy^{-1})^{x^2} = (1728)(36)(45), \quad (xy^{-1})^{x^3} = (3847)(16)(25),$

$(xy^{-1})^{xy} = (3645)(17)(28), \quad (xy^{-1})^{x^3 y} = (3546)(18)(27),$

$(xy^{-1})^{xy^2} = (1526)(37)(48), \quad (xy^{-1})^{x^2 y} = (1625)(38)(45).$

Table 1. Number of elements in each conjugacy class

S/N	Classes	Number of elements
1	$C1$	1
2	$C2$	1
3	$C3$	2
4	$C4$	4
5	$C5$	4
6	$C6$	4
7	$C7$	4
8	$C8$	4
9	$C9$	8
10	$C10$	8
11	$C11$	8
12	$C12$	8
13	$C13$	8

From the enumeration above, a summary of conjugacy class is shown. The number of elements in each class was obtained by counting the number of disjoint cycles form by each class. In Table 1, there are thirteen conjugacy classes of group of order 2^6 and hence thirteen irreducible representations (characters). That is, the number of irreducible representations is equal to the number of conjugacy classes.

4. Conclusion

It has been shown that the number of conjugacy classes of group of order 2^6 can be obtained using the commutator group.

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