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A Certain Class of Function Analytic and Subordinate to the Modified Sigmoid Function

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Abstract

A certain class of functions, analytic and subordinate to the modified sigmoid function, is defined. Coefficient inequalities, Toeplitz, distortion, and Fekete-Szegö problems of this class were investigated. It was observed that the results obtained provide extensions to many known results in geometric function theory. Special cases of the results were equally highlighted.

1 Introduction

Denote by ϱ the class of all functions

$$\Upsilon(\xi) = \xi + \sum_{\hbar=2}^{\infty} a_{\hbar}\xi^{\hbar}, \tag{1}$$

analytic in the open unit disc $\Delta = \{\xi : |\xi| < 1\}$ and are normalized by $\Upsilon(0) = \Upsilon'(0) - 1 = 0$. Let S denote the subclass of functions of ϱ which are univalent in the open unit disk Δ and let C, S^* and K be the usual subclasses of S consisting of functions that are convex, starlike and close-to-convex with respect with the origin. A function $\Upsilon \in \varrho$ is said to be univalent in the unit disk Δ if Υ is analytic in Δ and if for $\xi_1, \xi_2 \in \Delta, \Upsilon(\xi_1) = \Upsilon(\xi_2)$ whenever $\xi_1 = \xi_2$.

A function Υ is called a Bazilevic function of type α if there exists a function $g \in S^*$ such that for each $\alpha > 0$ and real δ, the following condition holds:

$$\text{Re}\left[\Upsilon'(\xi)\left(\frac{\Upsilon(\xi)}{\xi}\right)^{\alpha+i\delta-1}\left(\frac{g(z)}{\xi}\right)^{-\alpha}\right] > 0,$$

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whenever $\xi \in \Delta$.

The class of Bazilevic functions of type α shall be denoted by $\mathbb{B}(\alpha)$ (see Singh [18], Mohsan [13]) and we shall use that notation in this work. Note that $\mathbb{B}(\alpha) \subset S$, $\mathbb{B}(1) = K_0$ and $\mathbb{B}(0) \subset S^*$. Properties of $\mathbb{B}_1(1)$ has been investigated by many researchers in recent years. Shell-Small [17] obtained sharp estimates for the moduli of the coefficients of $\mathbb{B}_1(1)$ and its Fekete-Szego functional. Thomas [20] obtained sharp bounds for the second Hankel determinant, the initial coefficients of the function $log(\frac{\Upsilon(\xi)}{\xi})$ including the initial coefficients of the inverse of Υ. Marjono [12] presented the result of Singh concerning sharp value for the coefficients of $\mathbb{B}_1(1), |a_1|, |a_2|, |a_3|$ and $|a_4|$ and gave the solution for the Fekete-Szego problem $|a_3 - \mu a_2^2|$ for any real and complex number μ for the coefficients of function in $\mathbb{B}_1(\alpha)$. Some years later, Sadaf *et al.* [16], introduced and studied a subclass of analytic function defined using the concept of Bazilevic and Janowski functions and various properties like coefficient estimates, Fekete-Szego type inequalities, arc length problem and growth rate of coefficient were investigated. Elumalai and Abbas [2] recently introduced two new subclasses of regular and bi-univalent functions using Laguerre polynomials where some new and interesting results were obtained. More results were also obtained in this line of study by Murat *et al.* [14] and Made Asih *et al.* [10]. Fitri [5] obtain some coefficient inequalities of Bazilevic functions in a given sector. Using the idea of quasi-subordination, Karthikean *et al.* [7] studied the Bazilevic functions and obtained some results. In this work, it is aimed that some coefficient bounds, distortion and Fekete-Szego problems of the Bazilevic function related to the modified sigmoid function shall be investigated.

For $\Upsilon \in \varrho$, such that $\Upsilon(0) = 1$ and $\Upsilon'(0) = 0$, $\Upsilon \in \mathbb{B}_1(\alpha) \subset S$ if and only if

$$Re\Big\{\big[\Upsilon'(\xi)\frac{\Upsilon(\xi)}{\xi}\big]^{\alpha-1}\Big\} > 0, \xi \in \Delta, \quad \text{for} \quad \alpha \geq 0.$$

They are called the Bazivelic functions, in this case of order α.

An analytic function Υ is said to be subordinate to an analytic function g written as $\Upsilon(\xi) \prec g(\xi)$, if and only if there exists a function $w \in S$ of the form

$$w(\xi) = c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \cdots, (\xi \in \Delta),$$

analytic in Δ, such that $w(0) = 0, |w(\xi)| < 1$ and $\Upsilon(\xi) = g(w(\xi))$. If g is univalent in Δ, then $\Upsilon \prec g$ is equivalent to $\Upsilon(0) = g(0)$ and $g(\Delta) \subset \Upsilon(\Delta)$.

Pommerenke in 1975, defined for given parameters $q, \hbar \in \mathbb{N} = \{1, 2, 3, ...\}$, the Hankel determinant $Han_{\eta,\hbar}(\Upsilon)$ for a function $\Upsilon \in \varrho$ of the form (1) by

$$Han_{\eta,\hbar}(\Upsilon) = \begin{vmatrix} a_\hbar & a_{\hbar+1} & \cdots & a_{\hbar+\eta-1} \\ a_{\hbar+1} & a_{\hbar+2} & \cdots & a_{\hbar+\eta} \\ \vdots & \vdots & \cdots & \vdots \\ a_{\hbar+\eta-1} & a_{\hbar+\eta} & \cdots & a_{\hbar+2\eta-2}. \end{vmatrix}. \tag{2}$$

Thomas and Halim[19], defined the symmetric Toeplitz determinant $T_q(\hbar)$ for $q \geq 1$ as:

$$T_\eta(\hbar) = \begin{vmatrix} a_\hbar & a_{\hbar+1} & \cdots & a_{\hbar+\eta-1} \\ a_{\hbar+1} & a_{\hbar+2} & \cdots & a_{\hbar+\eta} \\ \vdots & \vdots & \cdots & \vdots \\ a_{\hbar+\eta-1} & a_{\hbar+\eta} & \cdots & a_{\hbar+2\eta-2}. \end{vmatrix}. \tag{3}$$

In Geometric Function Theory (GFT), various subclasses of S have been studied by many researchers. The study of univalent function theory dates back as far as twentieth century with many results obtained and the study is still on till date.

The following assertions are true; for $\Upsilon(\xi) = \xi + \sum_{\hbar=2}^{\infty} a_\hbar \xi^\hbar$; $\quad \Upsilon'(\xi) = 1 + \sum_{\hbar=2}^{\infty} \hbar a_\hbar \xi^{\hbar-1}$

$$\frac{\Upsilon(\xi)}{\xi} = 1 + \sum_{\hbar=2}^{\infty} a_\hbar \xi^{\hbar-1}$$

and

$$\begin{aligned} \Upsilon'(\xi)\Big[\frac{\Upsilon(\xi)}{\xi}\Big]^{\alpha-1} &= 1 + \sum_{\hbar=2}^{\infty} \hbar a_\hbar \xi^{\hbar-1} \cdot \Big\{1 + \sum_{\hbar=2}^{\infty} a_\hbar \xi^{\hbar-1}\Big\}^{\alpha-1} \\ &= 1 + (\alpha+1)a_2 z + \big[(\alpha+2)a_3 + \frac{(\alpha^2 - 3\alpha - 6)}{2} a_2^2\big]\xi^2 \\ &+ \frac{\alpha+3}{6}\big[(\alpha-1)(\alpha-2)a_2^3 + 6a_4 + (\alpha-1)a_2 a_3\big]\xi^3 \\ &+ (\alpha-1)\big[6a_4 a_2 + a_4 + \frac{\alpha-2}{2}(2a_2 a_4 + a_3^2) + (\alpha-2)(\frac{\alpha+1}{2})a_2^2 a_3 \\ &+ (\alpha-2)(\alpha-3)(\frac{3\alpha-1}{3})a_2^4 + 3a_3^2\big]\xi^4 + ... \end{aligned} \tag{4}$$

Again,

$$G(\xi) = 1 + \frac{1}{2}\xi - \frac{1}{24}\xi^3 + \frac{1}{240}\xi^5 - \frac{17}{40320}\xi^7 + ..,$$

(the series modified sigmoid function, Fadipe-Joseph *et al.* [4] and Ezeafulukwe *et al.* [3])

$$w(\xi) = c_1\xi + c_2\xi^2 + c_3\xi^3 + ..., \xi \in \triangle$$

$$w^3(\xi) = c_1^3\xi^3 + 3c_1^2 c_2 \xi^4 + (2c_1^2 c_3 + 3c_1 c_2^3)\xi^5 + ...$$

$$w^5(\xi) = c_1^5\xi^5 + ...$$

So that

$$G(w(\xi)) = 1 + \frac{1}{2}w(\xi) - \frac{1}{24}(w(\xi))^3 + \frac{1}{240}(w(\xi))^5 - \frac{17}{40320}(w(\xi))^7 + ...$$

After some mathematical substitutions and further simplifications, we obtain that

$$G(w(\xi)) = 1 + \frac{c_1}{2}\xi + \frac{c_2}{2}\xi^2 + (\frac{c_3}{2} - \frac{c_1^3}{24})\xi^3 + (\frac{c_4}{2} - \frac{c_1^2 c_2}{8})\xi^4 + ... \quad (5)$$

Now, let us give a definition of a class of Bazilevic functions related to the modified sigmoid function in the unit disc $\triangle$.

Definition 1.1. *Let $\Upsilon \in S$ be given by (1). For $\alpha \geq 0$ and $\xi \in \Delta$, we define the class $M^*_{\alpha,\hbar}(\xi)$ consisting of functions Υ that are analytic in the unit disk Δ and satisfy the following subordination condition:*

$$M^*_{\alpha,\hbar}(\xi) = \left\{ \Upsilon \in S : \Upsilon'(\xi) \cdot \frac{(\Upsilon(\xi))^{\alpha-1}}{\xi^{\alpha-1}} \prec G(\xi) \right\}, \quad (6)$$

where $\hbar \in \mathbb{N}$ and $G(\xi)$ is a modified sigmoid function.

Lemma 1.1 (Jahangiri [6]). *Let $p \in P$ and suppose $p(\xi) = 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots$. Then,*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 1.2 (Duren [1], Marjono [12]). *Let $p \in P$ be analytic in $\triangle$, and suppose*

$$p(\xi) = 1 + \sum_{\hbar=1}^{\infty} p_\hbar \xi^\hbar, \quad \hbar \in \mathbb{N}.$$

Then,

$$|p_\hbar| \leq 2.$$

This result is known as the Carathéodory-Toeplitz inequality, particularly for the extremal function $p(\xi) = \frac{1+\xi}{1-\xi}$.

Lemma 1.3 (Shi *et al.* [8]). *Let P denote the family of all functions p that are analytic in U with $R(p(\xi)) > 0$ and represented as*

$$p(\xi) = 1 + \sum_{\hbar=1}^{\infty} c_\hbar \xi^\hbar, \quad \xi \in \triangle .$$

Then the following inequalities hold:

- $|c_\hbar| \leq 2$ *for* $\hbar \geq 1$;
- $|c_{\hbar+k} - \mu c_\hbar c_k| < 2$ *for* $0 \leq \mu \leq 1$;
- $|c_m c_\hbar - c_k c_l| \leq 4$ *for* $m + \hbar = k + l$;
- $\left|c_{\hbar+2k} - \mu c_\hbar c_k^2\right| \leq 2(1 - 2\mu)$ *for* $\mu \in \mathbb{R}$;

- $\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{|c_1|^2}{2}$;
- *For any complex number* λ,

$$\left|c_2 - \lambda c_1^2\right| \le 2\max\{1, |2\lambda - 1|\}.$$

Lemma 1.4 (Libera [9])**.** *For* $p \in P$*, there exist complex numbers* x *and* ζ *with* $|x| \le 1$ *and* $|\zeta| \le 1$ *such that*

$$2c_2 = c_1^2 + x(4 + c_1^2),$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\zeta.$$

2 Main Results

Theorem 2.1. *Let* $\Upsilon \in M^*_{\alpha,\hbar}(\xi)$ *be given by* (1)*. Then:*

(i) $|a_2| \le \dfrac{1}{\alpha+1}, \quad \alpha \ge 0;$

(ii) $|a_3| \le \dfrac{\alpha^3 + 3\alpha^2 + 8\alpha + 1}{2(\alpha+1)^2(\alpha+2)};$

(iii) $|a_4| \le \dfrac{\alpha^2 + 7\alpha - 4}{2(\alpha+1)^2(\alpha+2)}.$

Proof. Equating coefficients in (4) and (5), we obtain that

$$(\alpha+1)a_2 = \frac{c_1}{2}$$

which implies that

$$a_2 = \frac{c_1}{2(\alpha+1)}$$

by Lemma (1.2), we obtain that

$$|a_2| \le \frac{1}{\alpha+1}.$$

Also,

$$(\alpha+2)a_3 + \frac{(\alpha^2 - 3\alpha + 6)}{2}a_2^2 = \frac{c_2}{2},$$

that is

$$a_3 = \frac{c_2}{\alpha+2} - \frac{c_1^2(\alpha^2 - 3\alpha + 6)}{8(\alpha+2)(\alpha+1)^2},$$

simplifying and applying Lemma (1.2) gives

$$|a_3| \le \frac{\alpha^3 + 3\alpha^2 + 8\alpha + 1}{2(\alpha+1)^2(\alpha+2)}.$$

We also have from (4) and (5) that

$$a_4=\frac{1}{6}\Big[\frac{12c_3-c_1^3}{4(\alpha+3)}-(\alpha-1)\Big[(\alpha+2)(\frac{c_1^3}{8(\alpha+1)})+\frac{c_1}{2(\alpha+1)}\frac{\alpha^2+7\alpha-4}{(\alpha+1)^3}\Big]\Big],$$

simplifying and using Lemma (1.3), we have

$$|a_4|\leq\frac{\alpha^2+7\alpha-4}{2(\alpha+1)^2(\alpha+2)}.$$

□

Corollary 2.2. *Let* $\Upsilon\in M^*_{\alpha,\hbar}(\xi)$ *be given by* (1)*. Then, for* $\alpha=1$*, we have*

$$|a_2|\leq\frac{1}{2},\quad |a_3|\leq\frac{13}{24},\quad |a_4|\leq\frac{1}{37}.$$

Theorem 2.3. *Let* $\Upsilon\in M^*_{\alpha,\hbar}(\xi)$ *be given by* (1)*. Then*

$$H_2(2)\leq\frac{\alpha^4+14\alpha^3+41\alpha^2-56\alpha+16}{4\alpha^6+32\alpha^5+104\alpha^4+184\alpha^3+164\alpha^2+80\alpha+16}.$$

Proof. Using the ideas first developed in (4) and (5) above, we have that

$$H_2(2)=|a_2a_4-a_3^2|\frac{c_1}{2(\alpha+1)}\frac{1}{6}\Big[\frac{12c_3-c_1^3}{4(\alpha+3)}-(\alpha+1)\Big[(\alpha-2)\frac{c_1^3}{8(\alpha+1)}\frac{\alpha^2+7\alpha-4}{(\alpha+1)^3}\Big]\Big]-\Big(\frac{\alpha^2+7\alpha-4}{2(\alpha^3+4\alpha^2+5\alpha+2)}\Big)^2.$$

Applying Lemma (1.4), we have that

$$\begin{aligned}H_2(2)&=\frac{3c_1(c_1^3+2(4-c_1^2)c_1x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)\zeta^2)}{2(\alpha+1)(24\alpha+18)}-\frac{(\alpha-2)c_1^4}{2(48\alpha+48)}\\&-\frac{c_1^2(\alpha^2+7\alpha-4)}{12(\alpha+1)^2}-\frac{\alpha^4+14\alpha^3+41\alpha^2-56\alpha+16}{4\alpha^6+32\alpha^5+184\alpha^4+184\alpha^3+164\alpha^2+80\alpha+16}.\end{aligned}\tag{7}$$

Noting that $h_2(2)$ is rotationally invariant and writing $c_1=c$, such that $0\leq c\leq 2$, we have

$$H_2(2)=\frac{3c_1(c_1^3+2(4-c_1^2)c_1x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)\zeta^2)}{2(\alpha+1)(24\alpha+18)}-\frac{(\alpha-2)c_1^4}{2(48\alpha+48)}$$

$$-\frac{c_1^2(\alpha^2+7\alpha-4)}{12(\alpha+1)^2}-\frac{\alpha^4+14\alpha^3+41\alpha^2-56\alpha+16}{4\alpha^6+32\alpha^5+184\alpha^4+184\alpha^3+164\alpha^2+80\alpha+16}.$$

Arbitrarily choosing $c=0$,

$$H_2(2)=-\Big(\frac{\alpha^4+14\alpha^3+41\alpha^2-56\alpha+16}{4\alpha^6+32\alpha^5+184\alpha^4+184\alpha^3+164\alpha^2+80\alpha+16}\Big).$$

So,

$$H_2(2)=|a_2a_4-a_3^2|\leq\frac{\alpha^4+14\alpha^3+41\alpha^2-56\alpha+16}{4\alpha^6+32\alpha^5+104\alpha^4+184\alpha^3+164\alpha^2+80\alpha+16}.$$

□

Theorem 2.4. *Let* $\Upsilon = \Upsilon(\xi) = \xi + \sum_{\hbar=2}^{\infty} a_{\hbar}\xi^{\hbar} \in M^*_{\alpha,\hbar}(\xi)$. *Then,* $|a_2 - \mu a_2^2| \leq \frac{2}{\alpha+2} + \frac{3\alpha-\alpha^2-68\mu\alpha-16\mu}{2(\alpha+2)(\alpha+1)^2}$.

Proof. Using a_2 and a_3 from (4) and Lemma (1.2), we have

$$|a_3 - \mu a_2^2| = \Big|\frac{c_2}{\alpha+2} - \frac{c_1^2(\alpha^2-3\alpha+6)}{8(\alpha+2)(\alpha+1)^2} - \mu\frac{c_1^2}{4(\alpha+1)^2}\Big|$$

$$= \Big|\frac{c_2}{\alpha+2} + c_1^2\Big(\frac{3\alpha-\alpha^2-6}{8(\alpha+2)(\alpha+1)^2} - \frac{\mu}{4(\alpha+1)^2}\Big)\Big|$$

$$|a_3 - \mu a_2^2| \leq \frac{|c_2|}{\alpha+2} + |c_1|^2\frac{3\alpha-\alpha^2-6}{8(\alpha+2)(\alpha+1)^2} - \frac{\mu}{4(\alpha+1)^2} \tag{8}$$

Application of Lemma (1.3) in (8), the result follows. □

Theorem 2.5. *For* $\Upsilon \in M^*_{\alpha,\hbar}(\xi), \alpha \geq 0$, *we have* $T_2(1) = |a_1^2 - a_2^2| \leq 1$.

Proof. The second Toeplitz determinant $T_2(1)$ for a function $\Upsilon \in \Im$ of the form (1) is given by

$$T_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix} = (a_1^2 - a_2^2) = \Big|1 - \frac{c_1^2}{4(\alpha+1)^2}\Big|. \tag{9}$$

Observe that $\Big(1 - \frac{c_1^2}{4(\alpha+1)^2}\Big) \geq 0$, since $0 \leq c_1 \leq 2$, and $\alpha > 0$. So, we have

$$T_2(1) = 1 - \frac{c_1^2}{4(\alpha+1)^2} := \varphi(c_1),$$

$$T_2'(1) = -\frac{2c_1}{4(\alpha+1)^2} \quad \forall \quad c_1 \in [0,2],\ \alpha > 0.$$

Observe that $\varphi(c_1)$ is a monotone decreasing function, so the maximum value is $\varphi(0) = 1$. The boundary inequality is sharp for $c_1 = 0$. Hence, the proof. □

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