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Some Results for Third-Order Differential Subordination and Superordination Involving the Fractional Derivative and Differential Operator

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Abstract

In the present paper, we define a certain suitable classes of admissible functions in the open unit disk associated with fractional derivative and differential operator. We derive some third-order subordination and superordination results for these classes. These results are applied to obtain third-order differential sandwich results. In addition, we indicate certain special cases and consequences for our results.

1. Introduction

Indicate by $\mathcal{H}(U)$ the collection of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \ \ a \in \mathbb{C}.$$

Also, let $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H}_1 = [1,1]$. Let $f, g \in \mathcal{H}(U)$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z) (z \in U)$. It is well known that, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [18]).

Let $\mathcal{A}$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1}$$

Significant and interesting problems in the geometric function theory are studied using third-order differential subordination and superordination for functions, which are analytic in the unit disk. In 1935, Goluzin [10] considered the simple first-order differential subordination $z\acute{p}(z) \prec h(z)$ and showed that if h is convex, then $p(z) \prec q(z) = \int_0^z h(t)\, t^{-1} dt$, and this q is the best dominant. In 1970, Suffridge [24] improved the Goluzin‘s result. In 1947, Robinson [21] considered the differential subordination $p(z) + z\acute{p}(z) \prec h(z)$ and showed that if h and $q(z) = z^{-1} \int_0^z h(z)\, dt$ are univalent, then q is the best dominant, at least for $|z| < 1/5$. In

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1975, Hallenbeck and Rusheweyh [11] considered the differential subordination:

$$p(z) + \frac{z\acute{p}(z)}{\gamma} \prec h(z) \quad (\gamma \neq 0, Re\gamma \geq 0)$$

and proved that if h is convex, then $p(z) \prec q(z) = \gamma z^{-\gamma} \int_0^z h(t)\, t^{\gamma-1} dt$, and this is the best dominant.

The theory of differential subordination in $\mathbb{C}$ is the complex analogue of differential inequality in $\mathbb{R}$. This theory of differential subordination was initiated by the works of Miller and Mocanu in 1981 [16], which was developed in other studies in 1987 [15] and 1989 [19]. Many significant works on differential subordination were pioneered by Miller and Mocanu, and their monograph (2000) [18] compiled their considerable efforts in introducing and developing the same. In 2003, Miller and Mocanu [17] investigated the dual problem of differential superordination, whereas Bulboaca (2005) [6] investigated both subordination and superordination. The theory of first and second order differential subordination and superordination has been used by numerous authors to solve problems in this field (see [1,2,4,8,14,28]). By contrast, few articles mentioned third-order inequalities and subordination. The first authors investigated the third order, and Ponnusamy et al. [19] published in 1992. In 2011, Antonino and Miller [3] extended the theory of second order differential subordination in the open unit disk U introduced by Miller Mocanu [18] to the third order case. They determined the properties of p functions that satisfy the following third-order differential subordination: $\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\} \subset \Omega$. In 2013, Jeyaraman et al. [13] also applied the third-order subordination result on the Schwarzian derivative. In 2014, Tang et al. [26] introduced the concept of the third-order differential superordination, which is a generalization of the second-order differential superordination. They determined the properties of function that satisfy the following third-order differential superordination:

$$\Omega \subset \{\psi(p(z), z\acute{p}(z), z^2p''(z), z^3p'''(z); z) : z \in U\}.$$

They also obtained the differential subordination and the corresponding differential superordination implications for meromorphically multivalent functions, which are defined by convolution operators involving the Liu-Srivastava operator by determining certain classes of admissible functions. In 2014, Tang et al. [25] investigated some third-order differential subordination result for analytic functions involving the generalized Bessel functions, in 2014, Tang et al. [27] studied the differential superordination based on analytic functions involving the generalized Bessel functions. In 2014, Farzana et al. [9] discussed some third-order differential subordination results for analytic functions which are associated with the fractional derivative operator.

This study used the methods of the third-order differential subordination and superordination results of Antonino and Miller [3] and Tang et al. [26], respectively. Certain suitable classes of admissible functions are considered in this study, and some applications of the third-order differential subordination and superordination of analytic functions associated with fractional derivative and differential operator. Several interesting examples are also discussed.

Definition 1.1 [20]. For $f \in \mathcal{A}$ the operator $I^{\eta,m}_{\lambda_1,\lambda_2,\ell,d}$ is defined by $I^{\eta,m}_{\lambda_1,\lambda_2,\ell,d} : \mathcal{A} \longrightarrow \mathcal{A}$

$$I^{\eta,m}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{M}^{m}_{\lambda_1,\lambda_2,\ell,d}(z) * R^{\eta} f(z), \quad z \in U,$$

where

$$\mathcal{M}^{m}_{\lambda_1,\lambda_2,\ell,d}(z) = z + \sum_{n=2}^{\infty} \left[\frac{\ell\big(1 + (\lambda_1 + \lambda_2)(n-1)\big) + d}{\ell\big(1 + \lambda_2(n-1)\big) + d}\right]^m z^n,$$

and $R^{\eta} f(z)$ denotes the Ruscheweyh derivative operator [22] given by

$$R^{\eta}f(z)=z+\sum_{n=2}^{\infty}C(\eta,n)a_nz^n\,,$$

where $C(\eta,n)=\binom{n+\eta-1}{\eta}, \eta, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \lambda_2\geq\lambda_1\geq 0\,, \ell\geq 0$ and $\ell+d>0$.

If f is given by (1.1), then we easily find that

$$I_{\lambda_1,\lambda_2,\ell,d}^{\eta,m}f(z)=z+\sum_{n=2}^{\infty}\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(n-1)\big)+d}{\ell\big(1+\lambda_2(n-1)\big)+d}\right]^m\binom{n+\eta-1}{\eta}a_nz^n. \tag{1.2}$$

Definition 1.2 [23]**.** The fractional derivative of order $\delta, (0\leq\delta<1)$ of a function f is defined by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\delta}}dt,$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-t)$ to be real, when $Re(z-t)>0$.

From Definition 1.1 and Definition 1.2, we have

$$D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{\eta,m}f(z)=\frac{1}{\Gamma(2-\delta)}z^{1-\delta}$$
$$+\sum_{n=2}^{\infty}\frac{n\Gamma(\eta+n)}{\Gamma(n-\delta+1)\Gamma(\eta+1)}\left[\frac{\ell\big(1+(\lambda_1+\lambda_2)(n-1)\big)+d}{\ell\big(1+\lambda_2(n-1)\big)+d}\right]^m a_nz^{n-\delta}. \tag{1.3}$$

It follows from (1.3) that

$$\ell\lambda_1z\left(D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{\eta,m}f(z)\right)'=\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{\eta,m+1}f(z)$$
$$-[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{\eta,m}f(z)\,. \tag{1.4}$$

Definition 1.3 [3]**.** Let $\psi:\mathbb{C}^4\times U\longrightarrow\mathbb{C}$. And the function $h(z)$ be univalent in U. If the function $p(z)$ is analytic in U and satisfies the following third-order differential subordination:

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\prec h(z), \tag{1.5}$$

then $p(z)$ is called a solution of the differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p(z)\prec q(z)$ for all $p(z)$ satisfying (5). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z)\prec q(z)$ for all dominants q of (1.5) is said to be the best dominant.

Definition 1.4 [3]**.** Let Q denote the set functions q that are analytic and univalent on the set $\overline{U}\setminus E(q)$, where

$$\mathrm{E}(q)=\left\{\xi\ :\xi\in\partial U:\lim_{z\to\xi}q(z)=\infty\right\},$$

and is such that $\min|q'(\xi)|=\rho>0$ for $\xi\in\partial U\setminus E(q)$.

Further, let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, $Q(0)\equiv Q_0$ and $Q(1)\equiv Q_1$.

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions was given by Antonino and Miller [3].

Definition 1.5 [3]**.** Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{1\}$. The family of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ achieving the following admissibility condition:

$$\phi(r, s, t, u; z) \notin \Omega,$$

whenever

$$r = q(\xi), \;\; s = k\xi q'(\xi) \;\text{ and }\; Re\left\{\frac{t}{s} + 1\right\} \geq k\, Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

and

$$Re\left\{\frac{u}{s}\right\} \geq k^2 Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\},$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq n$.

The next theorem is the foundation result in the theory of third-order differential subordination.

Theorem 1.1 [3]**.** *Let* $p \in \mathcal{H}[a, n]$ *with* $n \geq 2$, *and* $q \in \mathcal{Q}(a)$ *achieving the following conditions*:

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0, \quad \text{and} \qquad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq k,$$

where $z \in U, \xi \in \partial U \backslash E(q)$, *and* $k \geq n$. *If* Ω *is a set in* $\mathbb{C}, \psi \in \Psi_n[\Omega, q]$ *and*

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z) \qquad (z \in U).$$

Definition 1.6 [26]**.** Let $\psi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be analytic in U. If the function $p(z)$ and

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)$$

are univalent in U and satisfy the following third-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), \tag{1.6}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solution of the differential superordination, or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfy (1.6). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinant $q(z)$ of (1.6) is said to be the best subordinant.

Definition 1.7 [26]**.** Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The family of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(r, s, t, u; \xi) \in \Omega,$$

whenever

$$r = q(z), \;\; s = \frac{zq'(z)}{m} \;\text{ and }\; Re\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m}\, Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2} Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U,$ and $m \geq n \geq 2$.

Theorem 1.2 [9]. *Let* $q \in \mathcal{H}[a,n]$ *and* $\psi \in \acute{\Psi}_n[\Omega, q]$. *If*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$

is univalent in U *and* $p \in Q(a)$ *satisfy the following condition*:

$$Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(z)}\right| \leq m, \tag{1.7}$$

where $z \in U, \xi \in \partial U,$ *and* $m \geq n \geq 2$, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\}$$

implies that

$$q(z) \prec p(z) \qquad (z \in U).$$

2. Subordination Results

In this section, the following class of admissible functions is defined, which is required to prove the main third-order differential subordination theorem for the operator $D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)$ defined by (1.3).

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_D[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w, y; z) \notin \Omega$, whenever

$$u = q(\xi), \;\; v = \frac{\ell\lambda_1 \xi q'(\xi) + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]q(\xi)}{\left[\ell\left(1+\lambda_2(n-1)\right) + d\right]},$$

$$Re\left(\frac{x\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 v}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$
$$\left. - 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\right) \geq k\, Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$$Re\left(\frac{y\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3 v}{\ell^2{\lambda_1}^2(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$
$$- \frac{(2\ell\lambda_1 + (\ell\lambda_1 + 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])}{\ell\lambda_1}\left[\frac{x\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 v}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$
$$\left. - 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] - 1\right]$$
$$\left. - 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left(\frac{\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell^2{\lambda_1}^2}\right) - 1\right) \geq k^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U,\; \xi \in \partial U \backslash E(q)$ and $k \geq 2$.

Theorem 2.1. *Let* $\phi \in \Phi_D[\Omega, q]$. *If* $f \in \mathcal{A}$ *and* $q \in Q_0$ *satisfy the following conditions*:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{q'(\xi)}\right| \leq k, \tag{2.1}$$

and

$$\left\{\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) : z \in U\right\}, \tag{2.2}$$

then

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z), \quad (z \in U).$$

Proof. Define the function $p(z)$ in U by

$$p(z) = D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z). \tag{2.3}$$

From equations (1.4) and (2.3), we have

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z) = \frac{\ell\lambda_1 z p(z) + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]p(z)}{[\ell(1+\lambda_2(n-1))+d]}. \tag{2.4}$$

Further computations show that

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z) = \frac{\ell^2{\lambda_1}^2 z^2 p''(z) + \ell\lambda_1(\ell\lambda_1 + 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])zp'(z) + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 zp(z)}{[\ell(1+\lambda_2(n-1))+d]^2} \tag{2.5}$$

and

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z) = \frac{\begin{array}{c}\ell^3{\lambda_1}^3 z^3 p'''(z) + 3\ell^2{\lambda_1}^2(\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])z^2 p''(z) + \\ \{\ell\lambda_1(\ell^2{\lambda_1}^2 + 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])(\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])\}zp'(z) \\ +[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3\, p(z)\end{array}}{[\ell(1+\lambda_2(n-1))+d]^3}. \tag{2.6}$$

Define the transformation $\mathbb{C}^4$ to $\mathbb{C}$ by

$$v(r,s,t,u) = r, \quad w(r,s,t,u) = \frac{\ell\lambda_1 s + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]r}{[\ell(1+\lambda_2(n-1))+d]},$$

$$x(r,s,t,u) = \frac{\ell^2{\lambda_1}^2 t + \ell\lambda_1(\ell\lambda_1 + 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])s + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 r}{[\ell(1+\lambda_2(n-1))+d]^2}, \tag{2.7}$$

and

$$y(r,s,t,u) = \frac{\begin{array}{c}\ell^3{\lambda_1}^3 u + 3\ell^2{\lambda_1}^2(\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])t \\ + \{\ell\lambda_1(\ell^2{\lambda_1}^2 + 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])(\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])\}s \\ +[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3 r\end{array}}{[\ell(1+\lambda_2(n-1))+d]^3}. \tag{2.8}$$

Let

$$\psi(r,s,t,u;z) = \phi(v,w,x,y;z)$$

$$= \phi\left(\begin{array}{c} r, \dfrac{\ell\lambda_1 s + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]r}{[\ell(1+\lambda_2(n-1))+d]}, \dfrac{\ell^2{\lambda_1}^2 t + \ell\lambda_1(\ell\lambda_1 + 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])s + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 r}{[\ell(1+\lambda_2(n-1))+d]^2}, \\ \dfrac{\ell^3{\lambda_1}^3 u + 3\ell^2{\lambda_1}^2(\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])t + \{\ell\lambda_1(\ell^2{\lambda_1}^2 + 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])(\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])\}s + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3 r}{[\ell(1+\lambda_2(n-1))+d]^3}; z \end{array}\right). \tag{2.9}$$

The proof will make use of Theorem 1.1. Using equations (2.3) to (2.6), and from (2.9) we have

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$
$$= \phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right). \quad (2.10)$$

Hence, (2.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

Note that

$$\frac{t}{s} + 1 = \frac{x\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 v}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}$$
$$- 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]$$

and

$$\frac{u}{s} = \frac{y\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3 v}{\ell^2{\lambda_1}^2(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}$$
$$- \frac{(2\ell\lambda_1 + (\ell\lambda_1 + 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])}{\ell\lambda_1}$$

$$\left[\frac{x\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 v}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v} \atop -2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]-1\right]$$
$$- 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left(\frac{\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell^2{\lambda_1}^2}\right).$$

Thus, the admissibility condition for $\phi \in \Phi_D[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 1.5 with $n = 2$. Therefore, by using (2.2) and Theorem 1.1, we have

$$p(z) = D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 2.1. *Let $\Omega \in \mathbb{C}$ and let the function q be univalent in U with $q(0) = 0$. Let $\phi \in \Phi_D[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}$ and q_ρ satisfy the following conditions*:

$$Re\left(\frac{\xi {q''}_\rho(\xi)}{{q'}_\rho(\xi)}\right) \geq 0, \qquad \left|\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta m+1} f(z)}{{q'}_\rho(\xi)}\right| \leq k, \qquad \left(z \in U,\ \xi \in \partial U \backslash E(q_\rho)\right),$$

and

$$\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) \in \Omega.$$

then

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z) \qquad (z \in U).$$

Proof. Theorem 2.1 yields $D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z)(z \in U)$. The result asserted by Corollary 2.1 is now deduced from the following subordination property: $q_\rho(z) \prec q(z)$ $(z \in U)$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $p(z)$ of U onto Ω. In this case, the class $\Phi_D[p(U), q]$ is written as $\Phi_D[p, q]$. The following result follows immediately as a consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_D[h, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_0$ satisfy the following conditions* (2.1) *and*

$$\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) \prec p(z), \tag{2.11}$$

then

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z)(z \in U).$$

The next result is an immediate consequence of Corollary 2.1.

Corollary 2.2. *Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in U with $q(0) = 0$. Let $\phi \in \Phi_D[p, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}$ and q_ρ satisfy the following conditions*:

$$Re\left(\frac{\xi q''_\rho(\xi)}{q'_\rho(\xi)}\right) \geq 0, \quad \left|\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{q'_\rho(\xi)}\right| \leq k, \qquad \left(z \in U, \;\; \xi \in \partial U \backslash E(q_\rho)\right),$$

and

$$\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) \prec p(z),$$

then

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z)(z \in U).$$

The following result yields the best dominant of the differential subordination (2.11).

Theorem 2.3. *Let the function p be univalent in U and let $\phi : \mathbb{C}^4 \times u \to \mathbb{C}$ and ψ be given by* (2.9). *Suppose that the differential equation*

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) = h(z), \tag{2.12}$$

has a solution $q(z)$ with $q(0) = 0$ which satisfies condition (2.1). *If the function $f \in \mathcal{A}$ satisfies condition* (2.11) *and*

$$\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right)$$

is analytic in U, then

$$D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. From Theorem 2.1, we obtain q is a dominant of (2.11). Since q satisfies (2.12), it is also a solution of (2.11) and therefore q will be dominated by all dominants. Hence q is the best dominant.

In view of Definition 2.1, and in the special case $q(z) = Mz, M > 0$, the class of admissible functions $\Phi_D[\Omega, q]$, denoted by $\Phi_D[\Omega, M]$, is expressed as follows.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_D[\Omega, M]$, consists of those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\phi\Bigg(Me^{i\theta}, \frac{(k+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])Me^{i\theta}}{[\ell(1+\lambda_2(n-1))+d]},$$

$$\frac{L+(2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]K+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2)Me^{i\theta}}{[\ell(1+\lambda_2(n-1))+d]^2},$$

$$\{N+3\ell^2{\lambda_1}^2(\ell\lambda_1+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])L$$

$$+[\ell\lambda_1(\ell^2{\lambda_1}^2+3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])$$

$$(\ell\lambda_1+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])K+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3]Me^{i\theta}\}$$

$$\times\left(\left[\ell(1+\lambda_2(n-1))+d\right]^3\right)^{-1}; z\Bigg) \notin \Omega, \qquad (2.13)$$

whenever $z \in U, Re(Le^{-i\theta}) \geq (k-1)kM$, and $Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

Corollary 2.3. *Let $\phi \in \Phi_D[\Omega, M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z) \leq kM\right| \qquad (k \geq 2; M > 0),$$

and

$$\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) \in \Omega.$$

Then

$$\left|D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)\right| < M.$$

In the special case $\Omega = q(U) = \{w: |w| < M\}$, the class $\Phi_D[\Omega, M]$ is simply denoted by $\Phi_D[M]$. Corollary 2.4 can now be written in the following form:

Corollary 2.4. *Let $\phi \in \Phi_D[M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)\right| \leq kM \qquad (k \geq 2; M > 0),$$

and

$$\left|\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right)\right| < M,$$

then

$$\left|D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)\right| \leq M.$$

Example 2.1. Let $Re(m) \geq \frac{1-k}{2}, \ell(1+\lambda_2(n-1)) + d \neq 0, k \geq 2$ and $M > 0$. If the function $f \in \mathcal{A}$ satisfies

$$\left|D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)\right| < M,$$

then

$$\left|D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)\right| < M.$$

Proof. By taking $\phi(v,w,x,y;z)=w$ in Corollary 2.4, we have to find the condition so that $\phi \in \Phi_D[M]$, that is, the admissibility condition (2.13) is satisfied. This follows from

$$|\phi(v,w,x,y;z)| \geq M,$$

which implies

$$\left|\frac{(k+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])Me^{i\theta}}{\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]}\right| \geq M$$

or

$$|k+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]| \geq \left|\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]\right|. \tag{2.14}$$

Preceding inequality (2.14), shows that

$$Re(m) \geq \frac{1-k}{2}$$

Then it is sufficient to write

$$Re(m) \geq \frac{1-k}{2}$$

for (2.14) holds true. Hence, from Corollary 2.4, if $Re(m) \geq \frac{1-k}{2}, k \geq 2$ and

$$\left|D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)\right| < M, \text{then} \left|D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)\right| < M.$$

Example 2.2. Let $k \geq 2$ and $M > 0$. If the function $f \in \mathcal{A}$ satisfies

$$\left|D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)\right| \leq kM,$$

and

$$\left|D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z) - D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)\right| < \frac{M}{\left|\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]\right|},$$

then

$$\left|D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)\right| < M.$$

Proof. Let

$$\phi(v,w,x,y;z) = w - v, \quad \Omega = h(U),$$

where

$$h(z) = \frac{Mz}{\left|\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]\right|} \qquad (M > 0).$$

In order to use Corollary 2.3, we need to show that $\phi \in \Phi_D[\Omega, M]$, that is, the admissibility condition (2.13) is satisfied. This follows since

$$|\phi(v,w,x,y;z)| = \left|\frac{(k-1)Me^{i\theta}}{\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]}\right| \geq \frac{M}{\left|\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]\right|},$$

whenever $z \in U, \theta \in \mathbb{R}$ and $k \geq 2$. The required result now follows from Corollary 2.3.

Definition 2.3. Let Ω be a set in $\mathbb{C}, q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible function $\Phi_{D,1}[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(v, w, x, y; z) \notin \Omega$$

whenever

$$v = q(\xi), \qquad w = \frac{\ell\lambda_1 \xi q'(\xi) + [\ell\lambda_1(1-\delta) + \ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]q(\xi)}{\left[\ell\left(1+\lambda_2(n-1)\right) + d\right](1-\delta)!},$$

$$Re\left(\frac{\begin{array}{c} x(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2 - (1-\delta) \\ [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d](\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)v \end{array}}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v} - \frac{2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell\lambda_1} + 2(\delta-2)\right) \geq k\, Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$$Re\left(\frac{\dfrac{\begin{array}{c} y\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3(1-\delta)! - (1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] \\ (\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)(2\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v \end{array}}{\ell^2{\lambda_1}^2\left(w(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - (\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v)\right)}}{w(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - (\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v)} \right.$$

$$- \frac{\begin{array}{c} (\ell^2{\lambda_1}^2\left(6-\delta(7+2\delta)\right) + 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 + \ell\lambda_1(1-\delta) \\ [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] + 3(3-2\delta) \end{array}}{\ell^2{\lambda_1}^2(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3}$$

$$- \frac{(3\left[[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] + \ell\lambda_1(2-\delta)\right]}{\ell\lambda_1(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3}\left(\frac{x(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v)}\right.$$

$$- \frac{(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d](\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)v}{w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}$$

$$\left.\left. - \frac{2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell\lambda_1} - (3+2\delta)\right)\right) \geq k^2\, Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\},$$

where $z \in U, \xi \in \partial\mathcal{U} \setminus E(q)$ *and* $k \geq 2$.

Theorem 2.4. *Let* $\phi \in \Phi_{D,1}[\Omega, q]$. *If the function* $f \in \mathcal{A}$ *and* $q \in \mathcal{Q}_1$ *satisfy the following conditions*:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{zq'(\xi)}\right| \leq k, \tag{2.15}$$

and

$$\left\{\phi\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right) : z \in U\right\} \subset \Omega, \tag{2.16}$$

$then$

$$\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}} \prec q(z) \qquad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = \frac{(1-\delta)!\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}. \tag{2.17}$$

By using equation (1.4) and (2.17), we get

$$\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}} = \frac{(\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d])z^{1-\delta} p(z) + \ell\lambda_1 z^{2-\delta} p'(z)}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big) + d\right]}. \tag{2.18}$$

Further computations show that

$$\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}} = \frac{\begin{array}{c} \ell^2{\lambda_1}^2 z^{3-\delta} p''(z) + \ell\lambda_1\big(2[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + \ell\lambda_1(3-2\delta)\big) z^{2-\delta} p'(z) \\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] \\ \big(\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + 2\big) z^{1-\delta} p(z) \end{array}}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big) + d\right]^2} \tag{2.19}$$

and

$$\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}} = \frac{\begin{array}{c} \ell^3{\lambda_1}^3 z^{4-\delta} p'''(z) + \ell^2{\lambda_1}^2(3\{[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + \ell\lambda_1(2-\delta)\}) z^{3-\delta} p''(z) \\ +\ell\lambda_1 \left\{ \begin{array}{c} \ell^2{\lambda_1}^2\big(6 - \delta(7+2\delta)\big) + 2[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]^2 \\ +\ell\lambda_1[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]\left(\begin{array}{c} \big(\ell^2{\lambda_1}^2(1-\delta)^2 + 2[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]\big) \\ (1-\delta) + 3(3-2\delta) \end{array}\right) \end{array} \right\} z^{2-\delta} p'(z) \\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]\big\{\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + 2\big\} \times \\ \times\, (\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]) z^{1-\delta} p(z) \end{array}}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big) + d\right]^3}. \tag{2.20}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$v(r,s,t,u) = r,$$

$$w(r,s,t,u) = \frac{(\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d])r + \ell\lambda_1 s}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big) + d\right]}$$

$$x(r,s,t,u) = \frac{\begin{array}{c} \ell^2{\lambda_1}^2 t + \ell\lambda_1\big(2[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + \ell\lambda_1(3-2\delta)\big)s \\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] \\ \big(\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + 2\big)r \end{array}}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big) + d\right]^2} \tag{2.21}$$

and

$$y(r,s,t,u) = \frac{\begin{array}{c} \ell^3{\lambda_1}^3 u + \ell^2{\lambda_1}^2(3\{[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + \ell\lambda_1(2-\delta)\})t \\ +\ell\lambda_1 \left\{ \begin{array}{c} \ell^2{\lambda_1}^2\big(6 - \delta(7+2\delta)\big) + 2[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]^2 \\ +\ell\lambda_1[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]\left(\begin{array}{c} \big(\ell^2{\lambda_1}^2(1-\delta)^2 + 2[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]\big) \\ (1-\delta) + 3(3-2\delta) \end{array}\right) \end{array} \right\} s \\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]\big\{\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d] + 2\big\} \times \\ \times\, (\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d])r \end{array}}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big) + d\right]^3}. \tag{2.22}$$

Let

$$\psi(r,s,t,u;z)=\phi(v,w,x,y;z)$$

$$=\phi\left(\begin{array}{c} r,\dfrac{(\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])r+\ell\lambda_1 s}{(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]},\dfrac{\begin{array}{c}\ell^2{\lambda_1}^2t+\ell\lambda_1\left(2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(3-2\delta)\right)s\\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\\ \left(\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\right)r\end{array}}{(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2},\\ \dfrac{\begin{array}{c}\ell^3{\lambda_1}^3u+\ell^2{\lambda_1}^2(3\{[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(2-\delta)\})t\\ +\ell\lambda_1\left\{\begin{array}{c}\ell^2{\lambda_1}^2\left(6-\delta(7+2\delta)\right)+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2\\ +\ell\lambda_1[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left(\begin{array}{c}\left(\ell^2{\lambda_1}^2(1-\delta)^2+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\right)\\ (1-\delta)+3(3-2\delta)\end{array}\right)\end{array}\right\}s\\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left\{\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\right\}\\ (\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])r\end{array}}{(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3};z\end{array}\right) \tag{2.23}$$

The proof will make use of Theorem 1.1. Using equations (2.17) to (2.20), and from (2.23), we obtain

$$\begin{aligned}&\psi\left(z^{1-\delta}p(z),z^{2-\delta}p'(z),z^{3-\delta}p''(z),z^{4-\delta}p'''(z);z\right)\\ &=\phi\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m}f(z)}{z^{1-\delta}},\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1}f(z)}{z^{1-\delta}},\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2}f(z)}{z^{1-\delta}},\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3}f(z)}{z^{1-\delta}};z\right).\end{aligned} \tag{2.24}$$

Hence, (2.16) becomes

$$\psi\left(z^{1-\delta}p(z),z^{2-\delta}p'(z),z^{3-\delta}p''(z),z^{4-\delta}p'''(z);z\right)\in\Omega.$$

Note that

$$\frac{t}{s}+1=Re\left(\frac{\begin{array}{c}x(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^2-(1-\delta)\\ [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d](\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)v\end{array}}{\ell\lambda_1(w\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]-[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$

$$\left.-\frac{2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell\lambda_1}+2(\delta-2)\right)\geq\ k\,Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\}$$

$$\frac{u}{s}=\frac{\dfrac{\begin{array}{c}y\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3(1-\delta)!-(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\\ \left(\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\right)\\ (2\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v\end{array}}{\ell^2{\lambda_1}^2(w(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]-(\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v))}}{w(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]-(\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v)}$$

$$-\frac{\begin{array}{c}(\ell^2{\lambda_1}^2\left(6-\delta(7+2\delta)\right)+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2+\ell\lambda_1(1-\delta)\\ [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+3(3-2\delta)\end{array}}{\ell^2{\lambda_1}^2(1-\delta)!\left[\ell\left(1+\lambda_2(n-1)\right)+d\right]^3}$$

$$-\frac{(3\big[[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(2-\delta)\big]}{\ell\lambda_1(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^3}\left(\frac{x(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^2}{\ell\lambda_1\big(w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]-[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v\big)}\right.$$
$$-\frac{(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d](\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)v}{w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]-[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}$$
$$\left.-\frac{2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell\lambda_1}-(3+2\delta)\right).$$

Thus, the admissibility condition for $\phi\in\Phi_{D,1}[\Omega,q]$ in Definition 2.3, is equivalent to the admissibility condition for $\psi\in\Psi_2[\Omega,q]$ as given in Definition 1.5 with $n=2$. Therefore, by using (2.15) and Theorem 1.1, we have

$$p(z)=\frac{(1-\delta)!\,D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\prec q(z).$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_{D,1}[h(U),q]$. The following result follows immediately as a consequence of Theorem 2.4.

Theorem 2.5. *Let $\phi\in\Phi_{D,1}[\Omega,q]$. If the function $f\in\mathcal{A}$ and $q\in\mathcal{Q}_1$ satisfy the following conditions* (2.15) *and*

$$\phi\left(\frac{D_z^{\delta}I_{\lambda_1,\lambda_2}^{\eta,m}f(z)}{z^{1-\delta}},\frac{D_z^{\delta}I_{\lambda_1,\lambda_2}^{\eta,m+1}f(z)}{z^{1-\delta}},\frac{D_z^{\delta}I_{\lambda_1,\lambda_2}^{\eta,m+2}f(z)}{z^{1-\delta}},\frac{D_z^{\delta}I_{\lambda_1,\lambda_2}^{\eta,m+3}f(z)}{z^{1-\delta}};z\right)\prec h(z). \tag{2.25}$$

then

$$\frac{D_z^{\delta}I_{\lambda_1,\lambda_2}^{\eta,m}f(z)}{z^{1-\delta}}\prec q(z)\quad (z\in U).$$

In view of Definition 2.3, and in the special case $q(z)=Mz$, $M>0$, the class of admissible functions $\Phi_{D,1}[\Omega,q]$, is expressed as follows.

Definition 2.4. Let Ω be a set in $\mathbb{C}$ and $M>0$, the class of admissible functions $\Phi_{D,1}[\Omega,M]$ consists of those functions $\phi:\ \mathbb{C}^4\times U\longrightarrow\mathbb{C}$ such that

$$\phi\left(\begin{array}{c} Me^{i\theta},\dfrac{(\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])Me^{i\theta}+\ell\lambda_1K}{(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]},\dfrac{\begin{array}{c}\ell^2{\lambda_1}^2L+\ell\lambda_1\big(2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(3-2\delta)\big)s\\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\\ \big(\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\big)Me^{i\theta}\end{array}}{(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^2},\\ \dfrac{\begin{array}{c}\ell^3{\lambda_1}^3N+\ell^2{\lambda_1}^2(3\{[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(2-\delta)\})L\\ +\ell\lambda_1\left\{\begin{array}{c}\ell^2{\lambda_1}^2\big(6-\delta(7+2\delta)\big)+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2\\ +\ell\lambda_1[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left(\begin{array}{c}\big(\ell^2{\lambda_1}^2(1-\delta)^2+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\big)\\ (1-\delta)+3(3-2\delta)\end{array}\right)\end{array}\right\}K\\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\big\{\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\big\}\\ (\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])Me^{i\theta}\end{array}}{(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^3}\end{array}\right)\in\Omega, \tag{2.26}$$

whenever $z\in U, Re\big(Le^{-i\theta}\big)\geq(k-1)kM,$ and $Re\big(Ne^{-i\theta}\big)\geq 0$ for all $\theta\in\mathbb{R}$ and $k\geq 2.$

Corollary 2.5. *Let $\phi\in\Phi_{D,1}[\Omega,q]$. If the function $f\in\mathcal{A}$ satisfies*

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}\right| \leq kM \qquad (k \geq 2; M > 0),$$

and

$$\phi\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right) \in \Omega$$

then

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}\right| < M.$$

In the special case $\Omega = q(U) = \{w: |w| < M\}$, the class $\Phi_{D,1}[\Omega, q]$ is simple denoted by $\Phi_{D,1}[M]$. And Corollary 2.5, has the following from:

Corollary 2.6. *Let $\phi \in \Phi_D[M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}\right| \leq kM \qquad (k \geq 2; M > 0),$$

and

$$\left|\phi\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right)\right| < M,$$

then

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}\right| \leq M.$$

Example 2.3. Let $M > 0$. If the function $f \in \mathcal{A}$ satisfies

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}\right| < M,$$

then

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}\right| < M.$$

Proof. By taking $\phi(v, w, x, y; z) = w = \frac{(\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d])Me^{i\theta} + \ell\lambda_1 K}{(1-\delta)![\ell(1+\lambda_2(n-1)) + d]}$ in Corollary 2.6, the result is obtained.

Example 2.4. Let $M > 0$, and the function $f \in \mathcal{A}$ satisfies

$$\left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}\right| \leq kM,$$

and

$$\left|(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]^2\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2}f(z)}{z^{1-\delta}}-(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1}f(z)}{z^{1-\delta}}\right|$$
$$< M\left[2+\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\left(2+\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\right)\right],$$

then

$$\left|\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m}f(z)}{z^{1-\delta}}\right|<M.$$

Proof. Let

$$\phi(v,w,x,y;z)=(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]^2x+(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]w$$

and

$$\Omega=h(U),$$

where

$$h(z)=M\left[2+\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\left(2+\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\right)\right]z\quad (M>0).$$

In order to use Corollary 2.16, we need to show that $\phi\in\Phi_{D,1}[\Omega,M]$, that is, the admissibility condition (2.26) is satisfied. This follows since

$$\phi\left(\begin{array}{c} Me^{i\theta},\dfrac{(\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])Me^{i\theta}+\ell\lambda_1K}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]},\dfrac{\begin{array}{c}\ell^2{\lambda_1}^2L+\ell\lambda_1\big(2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(3-2\delta)\big)s\\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\\ \big(\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\big)Me^{i\theta}\end{array}}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]^2}, \\ \dfrac{\begin{array}{c}\ell^3{\lambda_1}^3N+\ell^2{\lambda_1}^2(3\{[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+\ell\lambda_1(2-\delta)\})L\\ +\ell\lambda_1\left\{\begin{array}{c}\ell^2{\lambda_1}^2\big(6-\delta(7+2\delta)\big)+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2\\ +\ell\lambda_1[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left(\begin{array}{c}\big(\ell^2{\lambda_1}^2(1-\delta)^2+2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\big)\\ (1-\delta)+3(3-2\delta)\end{array}\right)\end{array}\right\}K\\ +\ell\lambda_1(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\big\{\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2\big\}\\ (\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])Me^{i\theta}\end{array}}{(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]^3}\end{array}\right)\geq$$

$$\begin{aligned} Re\big(Le^{-i\theta}\big)&+(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\big((\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])\\ &+\ell\lambda_1K\big)M\\ &\geq k(k-1)M\\ &+(1-\delta)!\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\big((\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])\\ &+\ell\lambda_1K\big)M\geq M\left[2+\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\left(2+\left[\ell\big(1+\lambda_2(n-1)\big)+d\right]\right)\right],\end{aligned}$$

whenever $z\in U,\theta\in\mathbb{R}$ and $k\geq 2$.

3. Superordination of the Integral Operator $D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m}f(z)$

In this section, the third-order differential superordination theorem for the operator $D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m}f(z)$ defined by (1.3) is investigated. For this purpose, the class admissible function is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{H}_0$ with $q'(z)\neq 0$. The class of admissible functions $\Phi_D'[\Omega,q]$

consists of those functions $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w, y; z) \in \Omega,$$

whenever

$$u = q(z), \quad v = \frac{\ell\lambda_1 z q'(z) + m[\ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]q(z)}{m\big[\ell\big(1+\lambda_2(n-1)\big) + d\big]},$$

$$\mathrm{Re}\left(\frac{x\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^2 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 v}{\ell\lambda_1(w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v} - 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\right) \geq \frac{1}{m}\, Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$Re\left(\frac{y\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^3 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^3 v}{\ell^2{\lambda_1}^2(w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$
$$- \frac{(2\ell\lambda_1 + (\ell\lambda_1 + 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d])}{\ell\lambda_1}\left[\frac{x\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^2 - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 v}{\ell\lambda_1(w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$
$$\left. - 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] - 1\right]$$
$$\left. - 3[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\left(\frac{\ell\lambda_1 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell^2{\lambda_1}^2}\right) - 1\right) \geq \frac{1}{m^2} Re\left(\frac{z^2 q'''(z)}{q'(z)}\right).$$

where $z \in U,\ \xi \in \partial U \backslash E(q)$ and $m \geq 2$.

Theorem 3.1. *Let* $\phi \in \Phi_D^{'}[\Omega, q]$. *If* $f \in \mathcal{A}$ *and* $D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \in \mathcal{Q}_0$ *and* $q \in \mathcal{H}_0$ *with* $q'(z) \neq 0$ *satisfy the following conditions:*

$$Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta m+2} f(z)}{q'(z)}\right| \leq m,$$

and

$$\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right)$$

is univalent in U, *then*

$$\Omega \subset \left\{\phi\left(D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) : z \in U\right\} \tag{3.1}$$

implies that

$$q(z) \prec D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \qquad (z \in U).$$

Proof. Let the function $p(z)$ be defined by (2.3) and ψ by (2.9). Since $\phi \in \Phi_D^{'}[\Omega, q]$, (2.9) and (3.2) yield

$$\Omega \in \{\psi\,(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\}.$$

From equations (2.7) and (2.8), we see that the admissible condition for $\phi \in \Phi_D^{'}[\Omega, q]$ in Definition 3.1 is equivalent to the admissible condition for ψ as given in Definition 1.7 with $n = 2$. Hence $\psi \in {\Psi_2}'[\Omega, q]$, and by using (3.1) and Theorem 1.2, we have

$$q(z) \prec p(z) = D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U on to Ω. In this case, the class $\Phi'_D[h(U), q]$ is written as $\Phi'_D[h, q]$. The following result follows immediately as a consequence of Theorem 2.1.

Theorem 3.2. *Let* $\phi \in \Phi'_D[h, q]$ *and the function* h *be analytic in* U. *If the function* $f \in \mathcal{A}, I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \in \mathcal{Q}_0$ *and* $q \in \mathcal{H}_0$ *with* $q'(z) \neq 0$ *satisfy the following conditions* (3.1) *and*

$$\phi\left(D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right)$$

is univalent in U, *then*

$$h(z) \prec \phi\left(D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) \tag{3.2}$$

implies that

$$q(z) \prec D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \qquad (z \in U).$$

Theorem 3.1 and Theorem 3.2 can only be used to obtain subordinations of the third-order differential superordination of the forms (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for a suitable chosen ϕ.

Theorem 3.3. *Let the function* h *be analytic in* U *and let* $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ *and* ψ *be given by* (2.9). *Suppose that the differential equation*

$$\psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z\right) = h(z) \tag{3.3}$$

has a solution $q(z) \in \mathcal{Q}_0$. *If the function* $f \in \mathcal{A}$, $D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \in \mathcal{Q}_0$ *and* $q \in \mathcal{H}_0$ *with* $q'(z) \neq 0$ *satisfy the following conditions* (3.1) *and*

$$\phi\left(D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right)$$

is univalent in U, *then*

$$h(z) \prec \phi\left(D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right)$$

implies that

$$q(z) \prec D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \qquad (z \in U).$$

and q *is the best subordinant.*

Proof. In view of Theorem 3.1 and Theorem 3.2 we deduce that q is subordinant of (3.3).

Since q satisfies (3.3), it is also a solution of (3.2) and therefore q will be subordinants. Hence q is the best subordinant.

Combining Theorems 2.2 and 3.2, we obtain the following sandwich-type theorem.

Corollary 3.1. *Let* h_1 *and* q_1 *be analytic functions in* U, h_2 *be univalent function in* U, $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_D[h_2, q_2] \cap \Phi'_D[h_1, q_1]$. *If the function* $f \in \mathcal{A}$, $D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$, *and*

$$\phi\left(D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right),$$

is univalent in U, *and the conditions* (2.2) *and* (3.1) *are satisfied, then*

$$h_1(z) \prec \phi\left(D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z), D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z); z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z) \prec q_2(z) \qquad (z \in U).$$

Definition 3.2. Let Ω be a set in $\mathbb{C}$, and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible function $\Phi'_{D,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(v, w, x, y; \xi) \in \Omega$$

whenever

$$v = q(z), \qquad w = \frac{\ell\lambda_1 z q'(z) + m[\ell\lambda_1(1-\delta) + \ell(1+\lambda_2(n-1) - (1-\delta)\lambda_1) + d]q(z)}{m\big[\ell\big(1+\lambda_2(n-1)\big) + d\big](1-\delta)!},$$

and

$$Re\left(\frac{x(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^2 - (1-\delta)\,[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d](\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)v}{\ell\lambda_1(w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}\right.$$
$$\left. - \frac{2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell\lambda_1} + 2(\delta-2)\right) \geq \frac{1}{m}\, Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\}$$

$$Re\left(\frac{\dfrac{y\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^3(1-\delta)! - (1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]\,(\ell^2{\lambda_1}^2(1-\delta)^2+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)(2\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}{\ell^2{\lambda_1}^2(w(1-\delta)!\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]-(\ell\lambda_1(1-\delta)+[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v))}}{w(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - (\ell\lambda_1(1-\delta) + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v)}\right.$$
$$- \frac{(\ell^2{\lambda_1}^2\big(6-\delta(7+2\delta)\big) + 2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]^2 + \ell\lambda_1(1-\delta)\,[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] + 3(3-2\delta)}{\ell^2{\lambda_1}^2(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^3}$$
$$- \frac{(3\big[[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d] + \ell\lambda_1(2-\delta)\big]}{\ell\lambda_1(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^3}\left(\frac{x(1-\delta)!\,\big[\ell\big(1+\lambda_2(n-1)\big)+d\big]^2}{\ell\lambda_1(w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v)}\right.$$
$$- \frac{(1-\delta)[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d](\ell^2{\lambda_1}^2(1-\delta)^2 + [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]+2)v}{w\big[\ell\big(1+\lambda_2(n-1)\big)+d\big] - [\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]v}$$
$$\left.\left. - \frac{2[\ell(1+\lambda_2(n-1)-(1-\delta)\lambda_1)+d]}{\ell\lambda_1} - (3+2\delta)\right)\right) \geq \frac{1}{m^2}\, Re\left\{\frac{z^2 q'''(z)}{q'(z)}\right\},$$

where $z \in U, \xi \in \partial U \setminus E(q)$ and $m \geq 2$.

Theorem 3.4. *Let* $\phi \in \Phi'_{D,1}[\Omega, q]$. *If the function* $f \in \mathcal{A}$ *and* $\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}} \in Q_1$ *and* $q \in \mathcal{H}_1$ *with* $q'(z) \neq 0$ *satisfy the following conditions*:

$$Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{zq'(z)}\right| \leq m, \tag{3.4}$$

and

$$\phi\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right)$$

is univalent in U, *then*

$$\Omega \subset \left\{\phi\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right) : z \in U\right\}, \tag{3.5}$$

implies that

$$q(z) \prec \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}} \qquad (z \in U).$$

Proof. Let the function $p(z)$ be defined by (2.17) and ψ by (2.23). Since $\phi \in \Phi'_{D,1}[\Omega, q]$, (2.24) and (3.5) yield

$$\Omega \subset \{\psi\,(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\}.$$

From equations (2.21) and (2.22), we see that the admissible condition for $\phi \in \Phi'_D[\Omega, q]$ in Definition 3.2 is equivalent to the admissible condition for ψ as given in Definition 1.7 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega, q]$, and by using (3.4) and Theorem 1.2, we have

$$q(z) \prec p(z) = \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U on to Ω. In this case, the class $\Phi'_{D,1}[h(U), q]$ is written as $\Phi'_{D,1}[h, q]$. The following result follows immediately as a consequence of Theorem 3.4.

Theorem 3.5. *Let* $\phi \in \Phi'_{D,1}[h, q]$ *and the function* h *be analytic in* U. *If the function* $f \in \mathcal{A}$, $\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}} \in Q_1$ *and* $q \in \mathcal{H}_1$ *with* $q'(z) \neq 0$ *satisfy the following conditions* (3.5) *and*

$$\phi\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right)$$

is univalent in U, *then*

$$h(z) \prec \phi\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right)$$

implies that

$$q(z) \prec \frac{D_z^{\delta} I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \quad (z \in U).$$

Combining Theorem 2.5 and 3.5, we obtain the following sandwich-type theorem $\Phi'_{D,1}[\Omega, q]$.

Corollary 3.2. *Let* h_1 *and* q_1 *be analytic functions in* U, h_2 *be univalent function in* U, $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_{D,1}[h_2, q_2] \cap \Phi'_{D,1}[h_1, q_1]$. *If the function*

$$f \in \mathcal{A}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}} \in \mathcal{Q}_1 \cap \mathcal{H}_1,$$

and

$$\phi\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right)$$

is univalent in U, *and the condition* (2.13) *and* (3.5) *are satisfied, then*

$$h_1(z) \prec \phi\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+1} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+2} f(z)}{z^{1-\delta}}, \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m+3} f(z)}{z^{1-\delta}}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{D_z^\delta I_{\lambda_1,\lambda_2}^{\eta,m} f(z)}{Z^{1-\delta}} \prec q_2(z) \qquad (z \in U).$$

4. Conclusion

The primary objective was to use the fractional derivative in a complex domain and define a certain suitable classes of admissible functions in the open unit disk defined by differential operator. We considered several properties associated with third-order subordination and superordination to obtain a sandwich theorem. As future research directions, the contents of the paper on fractional derivative could in spire further research related to other classes.

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