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On Some Identities Related to Generalized Fibonacci and Lucas Numbers

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Abstract

In this paper, we define some new matrices similar to the classical matrices introduced by Gould in [9]. We calculate the nth powers of the new matrices by diagonalizing them with the help of eigenvalues and eigenvectors. Thus, by making use of Binomial expansions, we obtain new identities containing generalized Fibonacci and Lucas numbers. These new results inform us about the relationships between matrix algebra and sequence theory, especially in the context of generalized Fibonacci and Lucas sequences.

1 Introduction

Let k and t be distinct integers such that $k^2 + 4t \neq 0$. For $n \geq 2$, the generalized Fibonacci sequence (OEIS A015441), denoted by $(U_n(k,t))$, is defined as

$$U_0(k,t) = 0, U_1(k,t) = 1, U_n(k,t) = kU_{n-1}(k,t) + tU_{n-2}(k,t)$$

and the generalized Lucas sequence (OEIS A075117), denoted by $(V_n(k,t))$, is defined as

$$V_0(k,t) = 2, V_1(k,t) = P, V_n(k,t) = kV_{n-1}(k,t) + tV_{n-2}(k,t).$$

The characteristic equation for these sequences is $x^2 - kx - t = 0$, with roots given by

$$\alpha = \frac{k + \sqrt{k^2 + 4t}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 + 4t}}{2}.$$

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$U_n(k,t)$ and $V_n(k,t)$ are called nth generalized Fibonacci and Lucas numbers, respectively. Furthermore, for $n \in \mathbb{N}$, the negative-indexed generalized Fibonacci and Lucas numbers are defined as

$$U_{-n}(k,t) = -(-t)^{-n}U_n(k,t) \text{ and } V_{-n}(k,t) = (-t)^{-n}V_n(k,t). \tag{1}$$

These sequences are defined firstly by Lucas in [2]. When $k = t = 1$, these sequences reduce to the Fibonacci sequence (OEIS A000045) (F_n) and the Lucas sequence (OEIS A000032) (L_n), respectively. From now on, for simplicity we will write $U_n = U_n(k,t)$ and $V_n = V_n(k,t)$.

The Binet's formulas for the generalized Fibonacci and Lucas numbers are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n.$$

For any integer n, the identity

$$V_n = U_{n+1} + tU_{n-1} = kU_n + 2tU_{n-1} \tag{2}$$

is well known and can be proved by the Binet's formulas.

There exist numerous identities related to generalized Fibonacci and Lucas sequences in the literature. Matrices, mathematical induction, and Binet's formulas are frequently utilized to prove these identities. One of the most well-known matrices used to derive these identities is the Fibonacci Q matrix:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The nth power of the Q matrix is

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \text{ (see [3])}.$$

The Cassini identity stated by Robert Simson in 1753,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

is obtained using the equation $|Q|^n = |Q^n|$ associated with the matrix Q. The same identity is also derived from the nth power of the matrix

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

whose power is

$$R^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

In [5,6], the authors considered the matrix $M = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and demonstrated that

$$M^n = \begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}.$$

Here, using the equality $|M|^n = |M^n|$, the most general form of the Cassini identity

$$U_{n+1}U_{n-1} - U_n^2 = (-t)^{n-1}$$

is obtained. For more information and applications of these sequences one can consult [7] and [8], respectively.

In [9], taking $a \neq b$, Gould showed that nth power of the matrix

$$\begin{pmatrix} a-b & -ab \\ 1 & 0 \end{pmatrix}$$

is

$$\begin{pmatrix} \frac{a^{n+1}-b^{n+1}}{a-b} & \frac{-ab(a^n-b^n)}{a-b} \\ \frac{a^n-b^n}{a-b} & \frac{-ab(a^{n-1}-b^{n-1})}{a-b} \end{pmatrix}.$$

Inspired of this matrix, we give some other matrices whose powers consist of the generalized Fibonacci numbers and then we will give some identities by using those matrices. We think many of our identities are new in the literature.

2 Main Theorems

Theorem 1. *Let a and b be real numbers different from zero and each other. Then the following are true:*

M	M^n
$\begin{pmatrix} a+b & -a \\ b & 0 \end{pmatrix}$	$\frac{1}{a-b}\begin{pmatrix} a^{n+1}-b^{n+1} & -a(a^n-b^n) \\ b(a^n-b^n) & -ab(a^{n-1}-b^{n-1}) \end{pmatrix}$
$\begin{pmatrix} a+b & a \\ -b & 0 \end{pmatrix}$	$\frac{1}{a-b}\begin{pmatrix} a^{n+1}-b^{n+1} & a(a^n-b^n) \\ -b(a^n-b^n) & -ab(a^{n-1}-b^{n-1}) \end{pmatrix}$
$\begin{pmatrix} 0 & -a \\ b & a+b \end{pmatrix}$	$\frac{1}{a-b}\begin{pmatrix} -ab(a^{n-1}-b^{n-1}) & -a(a^n-b^n) \\ b(a^n-b^n) & a^{n+1}-b^{n+1} \end{pmatrix}$
$\begin{pmatrix} 0 & a \\ -b & a+b \end{pmatrix}$	$\frac{1}{a-b}\begin{pmatrix} -ab(a^{n-1}-b^{n-1}) & a(a^n-b^n) \\ -b(a^n-b^n) & a^{n+1}-b^{n+1} \end{pmatrix}$
$\begin{pmatrix} 0 & -b \\ a & a+b \end{pmatrix}$	$\frac{1}{a-b}\begin{pmatrix} -ab(a^{n-1}-b^{n-1}) & -b(a^n-b^n) \\ a(a^n-b^n) & a^{n+1}-b^{n+1} \end{pmatrix}$
$\begin{pmatrix} a+b & b \\ -a & 0 \end{pmatrix}$	$\frac{1}{a-b}\begin{pmatrix} a^{n+1}-b^{n+1} & b(a^n-b^n) \\ -a(a^n-b^n) & -ab(a^{n-1}-b^{n-1}) \end{pmatrix}$

Proof. The characteristic equation of matrix $M = \begin{pmatrix} a+b & -a \\ b & 0 \end{pmatrix}$ is $\lambda^2 - (a+b)\lambda + ab = 0$, and it can be easily seen that its roots are a and b. Since a and b are distinct eigenvalues of M, the matrix M can be diagonalized. Firstly, the eigenvectors corresponding to the eigenvalue a for the matrix M are obtained from the equation

$$\begin{pmatrix} b & -a \\ b & -a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Thus, we find that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} as \\ bs \end{pmatrix}$ where $s \neq 0$. When $s = 1$, we obtain $\begin{pmatrix} a \\ b \end{pmatrix}$ as an eigenvector. Similarly, for the eigenvector related to b, we find the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, we can take $P = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then, we have $P^{-1} = \frac{1}{a-b}\begin{pmatrix} 1 & -1 \\ -b & a \end{pmatrix}$ since $a - b \neq 0$. Since $M = PQP^{-1}$, it follows that $M^n = PQ^nP^{-1}$. Consequently, we find that

$$M^n = \frac{1}{a-b}\begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n\begin{pmatrix} 1 & -1 \\ -b & a \end{pmatrix} = \begin{pmatrix} \frac{a^{n+1}-b^{n+1}}{a-b} & \frac{-a(a^n-b^n)}{a-b} \\ \frac{b(a^n-b^n)}{a-b} & \frac{-ab(a^{n-1}-b^{n-1})}{a-b} \end{pmatrix}.$$

The proof are similar for the other matrices. □

If a is replaced by α and b is replaced by β, where $a+b=\alpha+\beta=k$, $-ab=-\alpha\beta=t$, then we get

$$\frac{a^n-b^n}{a-b}=U_n \text{ and } a^n+b^n=V_n.$$

Therefore, the following theorem can be given easily.

Corollary 2. *Let k and t be non-zero and distinct integers. Then the following identities are true.*

M	M^n
$\begin{pmatrix} k & -\alpha \\ \beta & 0 \end{pmatrix}$	$\begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha \\ -\beta & k \end{pmatrix}$	$\begin{pmatrix} tU_{n-1} & \alpha U_n \\ -\beta U_n & U_{n+1} \end{pmatrix}$
$\begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix}$	$\begin{pmatrix} tU_{n-1} & -\alpha U_n \\ \beta U_n & U_{n+1} \end{pmatrix}$
$\begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix}$	$\begin{pmatrix} U_{n+1} & \alpha U_n \\ -\beta U_n & tU_{n-1} \end{pmatrix}$

Theorem 3. *Let n be a natural number. Then*

$$\begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}^n = \begin{cases} (k^2+4t)^{\frac{n}{2}} I, & n \text{ is an even natural number} \\ (k^2+4t)^{\frac{n-1}{2}} \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}, & n \text{ is an odd natural number} \end{cases}.$$

Proof. Since $\alpha\beta=-t$, we get

$$\begin{aligned}\begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}^2 &= \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}\begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix} \\ &= \begin{pmatrix} k^2-4\alpha\beta & 0 \\ 0 & k^2-4\alpha\beta \end{pmatrix} \\ &= (k^2+4t)I.\end{aligned}$$

Then the proof follows. □

The proof of the following theorem is omitted because it is similar to the previous proof.

Theorem 4. *Let n be a natural number. Then*

$$\begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix}^n = \begin{cases} (k^2+4t)^{\frac{n}{2}} I, & n \text{ is an even natural number} \\ (k^2+4t)^{\frac{n-1}{2}} \begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix}, & n \text{ is an odd natural number} \end{cases}.$$

Theorem 5. *If n is an even natural number, then it follows that*

$$\begin{aligned} 0 &= \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j}, \\ (k^2+4t)^{\frac{n}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^{j+1} U_{n-2j-1}, \\ (k^2+4t)^{\frac{n}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j+1}, \\ 2(k^2+4t)^{\frac{n}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^j V_{n-2j}, \end{aligned}$$

and if n is an odd natural number, then it follows that

$$\begin{aligned} 0 &= \sum_{j=0}^{n} \binom{n}{j} t^j V_{n-2j}, \\ k(k^2+4t)^{\frac{n-1}{2}} &= -\sum_{j=0}^{n} \binom{n}{j} t^{j+1} U_{n-2j-1}, \\ k(k^2+4t)^{\frac{n-1}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j+1}, \\ 2(k^2+4t)^{\frac{n-1}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j}. \end{aligned}$$

Proof. Let $N = \begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix}$ and $M = \begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix}$. Then, we find that

$$MN = NM = -tI, N + M = kI, M - N = \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}.$$

Also,

$$M^n = \begin{pmatrix} U_{n+1} & \alpha U_n \\ -\beta U_n & tU_{n-1} \end{pmatrix},$$

$$(M-N)^n = \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}^n = (k^2+4t)^{n/2} I \text{ for even } n, \tag{3}$$

and

$$(M-N)^n = \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}^n = (k^2+4t)^{(n-1)/2} \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix} \text{ for odd } n, \tag{4}$$

by Corollary 2 and Theorem 3. Moreover,

$$(M-N)^n = \sum_{j=0}^{n} \binom{n}{n-j} M^{n-j}(-N)^j = \sum_{j=0}^{n} \binom{n}{n-j} M^{n-2j} t^j. \tag{5}$$

Therefore, if n is odd, then we obtain

$$\left(k^2+4t\right)^{\frac{n-1}{2}} \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j+1} & \alpha \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j} \\ -\beta \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j} & \sum_{j=0}^{n} \binom{n}{j} t^{j+1} U_{n-2j-1} \end{pmatrix}$$

by using (4) and (5) and if n is even, then we obtain

$$\left(k^2+4t\right)^{\frac{n}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j+1} & \alpha \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j} \\ -\beta \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j} & \sum_{j=0}^{n} \binom{n}{j} t^{j+1} U_{n-2j-1} \end{pmatrix}$$

by using (3) and (5). Additionally, by using (2), we get the results easily. □

Theorem 6. *If n is an odd natural number, then*

$$0 = \sum_{j=0}^{n-1} t^j U_{n-2j-1},$$
$$V_n = k \sum_{j=0}^{n-1} t^j U_{n-2j} = k \sum_{j=0}^{n-1} t^{j+1} U_{n-2j-2}.$$

Proof. Let $N = \begin{pmatrix} 0 & \alpha \\ -\beta & k \end{pmatrix}$ and $M = \begin{pmatrix} k & -\alpha \\ \beta & 0 \end{pmatrix}$. Then, $MN = NM = -tI$, $N+M = kI$. Also,

$$M^n = \begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix},$$

and

$$N^n = \begin{pmatrix} tU_{n-1} & \alpha U_n \\ -\beta U_n & U_{n+1} \end{pmatrix}.$$

Here,

$$M^n + N^n = \begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix} + \begin{pmatrix} tU_{n-1} & \alpha U_n \\ -\beta U_n & U_{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} V_n & 0 \\ 0 & V_n \end{pmatrix} = V_n I$$

by (2) and Corollary 2. We have

$$V_n I = M^n + N^n = (M+N)\sum_{j=0}^{n-1} M^{(n-1-j)}(-N)^j$$
$$= k\sum_{j=0}^{n-1} M^{n-1-2j} t^j$$

since n is odd. Therefore, it can be seen that

$$\begin{pmatrix} V_n & 0 \\ 0 & V_n \end{pmatrix} = \begin{pmatrix} k\sum_{j=0}^{n-1} t^j U_{n-2j} & -\alpha k \sum_{j=0}^{n-1} t^j U_{n-2j-1} \\ \beta k \sum_{j=0}^{n-1} t^j U_{n-2j-1} & kt\sum_{j=0}^{n-1} t^j U_{n-2j-2} \end{pmatrix}.$$

This proof is completed. □

Theorem 7. *For natural numbers m and n, the following are true:*

$$U_m U_{mn+mk+m} = U_{mn+m} U_{mk+m} + (-t)^m U_{mn} U_{mk},$$
$$U_m U_{mn+mk} = U_{mn+m} U_{mk} - (-t)^m U_{mn} U_{mk-m},$$
$$U_m U_{mn+mk-m} = -U_{mn} U_{mk} - (-t)^m U_{mn-m} U_{mk-m}.$$

Proof. Let $M = \begin{pmatrix} a+b & -a \\ b & 0 \end{pmatrix}$. If we substitute α^m for a and β^m for b, then we obtain the matrix $M = \begin{pmatrix} V_m & -\alpha^m \\ \beta^m & 0 \end{pmatrix}$. By using Theorem 1, we obtain

$$M^{n+k} = \begin{pmatrix} \frac{U_{mn+mk+m}}{U_m} & \frac{-\alpha^m U_{mn+mk}}{U_m} \\ \frac{-\beta^m U_{mn+mk}}{U_m} & \frac{-(-t)^m U_{mn+mk-m}}{U_m} \end{pmatrix} = M^n \cdot M^k$$
$$= \begin{pmatrix} \frac{U_{mn+m}}{U_m} & \frac{-\alpha^m U_{mn}}{U_m} \\ \frac{-\beta^m U_{mn}}{U_m} & \frac{-(-t)^m U_{mn-m}}{U_m} \end{pmatrix} \begin{pmatrix} \frac{U_{mk+m}}{U_m} & \frac{-\alpha^m U_{mk}}{U_m} \\ \frac{-\beta^m U_{mk}}{U_m} & \frac{-(-t)^m U_{mk-m}}{U_m} \end{pmatrix}.$$

The proof is completed from this equality. □

Theorem 8. *For any natural number n, the following hold true:*

$$k(k^2+4t)^n = \sum_{j=0}^{2n+1}\binom{2n+1}{j}t^jU_{2n-2j+2} = -\sum_{j=0}^{2n+1}\binom{2n+1}{j}t^{j+1}U_{2n-2j}, \tag{6}$$

$$0 = \sum_{j=0}^{2n+1}\binom{2n+1}{j}t^jV_{2n-2j}, \tag{7}$$

$$2(k^2+4t)^n = \sum_{j=0}^{2n+1}\binom{2n+1}{j}t^jU_{2n-2j+1}. \tag{8}$$

Proof. Let $N = \begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix}$ and $M = \begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix}$. Then

$$M - N = \begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix} = \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}, \tag{9}$$

$$MN = NM = -tI,$$

and

$$M^n = \begin{pmatrix} U_{n+1} & \alpha U_n \\ -\beta U_n & tU_{n-1} \end{pmatrix}. \tag{10}$$

From Theorem 3, we get

$$(M-N)^{2n+1} = ((M-N)^2)^n(M-N) = (k^2+4t)^n(M-N)$$
$$= (k^2+4t)^n\begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix}. \tag{11}$$

Also

$$(M-N)^{2n+1} = \sum_{j=0}^{2n+1}\binom{2n+1}{j}M^{2n+1-j}(-N)^j$$

and so

$$(k^2+4t)^n\begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix} = \sum_{j=0}^{2n+1}\binom{2n+1}{j}M^{2n+1-2j}t^j. \tag{12}$$

Therefore, by using (10), (11), and (12), we obtain

$$\begin{pmatrix} k(k^2+4t)^n & 2\alpha(k^2+4t)^n \\ -2\beta(k^2+4t)^n & -k(k^2+4t)^n \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{j=0}^{2n+1}\binom{2n+1}{j}t^jU_{2n-2j+2} & \alpha\sum_{j=0}^{2n+1}\binom{2n+1}{j}t^jU_{2n-2j+1} \\ -\beta\sum_{j=0}^{2n+1}\binom{2n+1}{j}t^jU_{2n-2j+1} & \sum_{j=0}^{2n+1}\binom{2n+1}{j}t^{j+1}U_{2n-2j} \end{pmatrix}.$$

Hence (6) and (8) are obtained from the matrix equality. By using (6) and (2), the equation (7) can be shown easily. □

Theorem 9. *Let m and n be natural numbers. The following holds:*

$$V_m^n = \sum_{j=0}^{n} \binom{n}{j} (-t)^{mj} U_{mn+1-2mj} = t \sum_{j=0}^{n} \binom{n}{j} (-t)^{mj} U_{mn-1-2mj}, \tag{13}$$

$$2V_m^n = \sum_{j=0}^{n} \binom{n}{j} (-t)^{mj} V_{mn-2mj}, \tag{14}$$

$$0 = \sum_{j=0}^{n} \binom{n}{j} (-t)^{mj} U_{mn-2mj}. \tag{15}$$

Proof. Let $M = \begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix}$. As $MN = NM = -tI$, then, from Corollary 2, it is clear that

$$M^m + N^m = \begin{pmatrix} U_{m+1} & \alpha U_m \\ -\beta U_m & tU_{m-1} \end{pmatrix} + \begin{pmatrix} tU_{m-1} & -\alpha U_m \\ \beta U_m & U_{m+1} \end{pmatrix} = \begin{pmatrix} V_m & 0 \\ 0 & V_m \end{pmatrix} = V_m I,$$

and therefore

$$\begin{aligned} V_m^n I = (V_m I)^n &= (M^m + N^m)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} (M^m)^{n-j} (N^m)^j \\ &= \sum_{j=0}^{n} \binom{n}{j} M^{mn-2mj} (-t)^{mj}. \end{aligned}$$

Since

$$V_m^n I = \begin{pmatrix} V_m^n & 0 \\ 0 & V_m^n \end{pmatrix},$$

we have

$$\begin{pmatrix} V_m^n & 0 \\ 0 & V_m^n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} U_{mn+1-2mj} (-t)^{mj} & \alpha \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj} (-t)^{mj} \\ -\beta \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj} (-t)^{mj} & t \sum_{j=0}^{n} \binom{n}{j} U_{mn-1-2mj} (-t)^{mj} \end{pmatrix}.$$

Then (13) and (15) can be easily derived. By using the identities (13) and (2), the identity (14) can be obtained. □

Theorem 10. *Let m and n be natural numbers. Then the following hold true:*

If n is an even natural number, then

$$U_m^n(k^2+4t)^{\frac{n}{2}} = t\sum_{j=0}^{n}\binom{n}{j}U_{mn-1-2mj}(-t)^{mj}(-1)^j \tag{16}$$

$$= \sum_{j=0}^{n}\binom{n}{j}U_{mn+1-2mj}(-t)^{mj}(-1)^j, \tag{17}$$

$$2U_m^n(k^2+4t)^{\frac{n}{2}} = \sum_{j=0}^{n}\binom{n}{j}(-1)^j(-t)^{mj}V_{mn-2mj}, \tag{18}$$

$$0 = \sum_{j=0}^{n}\binom{n}{j}U_{mn-2mj}(-t)^{mj}(-1)^j, \tag{19}$$

If n is an odd natural number, then

$$kU_m^n(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n}\binom{n}{j}(-t)^{mj}(-1)^jU_{mn+1-2mj} \tag{20}$$

$$= -t\sum_{j=0}^{n}\binom{n}{j}(-t)^{mj}(-1)^jU_{mn-1-2mj}, \tag{21}$$

$$2U_m^n(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n}\binom{n}{j}(-t)^{mj}(-1)^jU_{mn-2mj}, \tag{22}$$

$$0 = \sum_{j=0}^{n}\binom{n}{j}(-t)^{mj}(-1)^jV_{mn-2mj}. \tag{23}$$

Proof. Let $M = \begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix}$. As

$$MN = NM = -tI,\ M - N = \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix},$$

from Corollary 2 and Theorem 3, we have

$$M^m - N^m = \begin{pmatrix} U_{m+1} & \alpha U_m \\ -\beta U_m & tU_{m-1} \end{pmatrix} - \begin{pmatrix} tU_{m-1} & -\alpha U_m \\ \beta U_m & U_{m+1} \end{pmatrix} = \begin{pmatrix} kU_m & 2\alpha U_m \\ -2\beta U_m & -kU_m \end{pmatrix}$$
$$= U_m\begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix} = U_m(M-N)$$

and

$$U_m^n(M-N)^n = (U_m(M-N))^n = (M^m - N^m)^n = \sum_{j=0}^{n} \binom{n}{j} (M^m)^{n-j} (-N^m)^j$$
$$= \sum_{j=0}^{n} \binom{n}{j} M^{mn-2mj} (-t)^{mj} (-1)^j.$$

If n is an even natural number, from Theorem 3, then we get

$$U_m^n(M-N)^n = \begin{pmatrix} U_m^n(k^2+4t)^{\frac{n}{2}} & 0 \\ 0 & U_m^n(k^2+4t)^{\frac{n}{2}} \end{pmatrix}.$$

Therefore,

$$U_m^n(k^2+4t)^{\frac{n}{2}} I = \begin{pmatrix} \sum_{j=0}^{n} U_{mn+1-2mj} w_j & \alpha \sum_{j=0}^{n} U_{mn-2mj} w_j \\ -\beta \sum_{j=0}^{n} U_{mn-2mj} w_j & t \sum_{j=0}^{n} U_{mn-1-2mj} w_j \end{pmatrix}$$

where $w_j = \binom{n}{j}(-t)^{mj}(-1)^j$. The identities (16), (17), and (19) follow easily. By using (16) and (17), the equation (18) can be shown. If n is an odd natural number, then from Theorem 3, we get

$$(M-N)^n = \begin{pmatrix} k(k^2+4t)^{\frac{n-1}{2}} & 2\alpha(k^2+4t)^{\frac{n-1}{2}} \\ -2\beta(k^2+4t)^{\frac{n-1}{2}} & -k(k^2+4t)^{\frac{n-1}{2}} \end{pmatrix}.$$

Therefore, we have

$$U_m^n(k^2+4t)^{\frac{n-1}{2}} \begin{pmatrix} k & 2\alpha \\ -2\beta & -k \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n} U_{mn+1-2mj} w_j & \alpha \sum_{j=0}^{n} U_{mn-2mj} w_j \\ -\beta \sum_{j=0}^{n} U_{mn-2mj} w_j & t \sum_{j=0}^{n} U_{mn-1-2mj} w_j \end{pmatrix}$$

where $w_j = \binom{n}{j}(-t)^{mj}(-1)^j$. Therefore, the identities (20), (21), and (22) are obtained. By using (2), (20) and (21), the equation (23) can be shown easily. □

Theorem 11. *For natural numbers m and n, the following are true:*

$$U_{mn} = -\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} U_{mn-mj} V_m^j, \tag{24}$$

$$U_{mn+1} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} t U_{mn-1-mj} V_m^j, \tag{25}$$

$$tU_{mn-1} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} U_{mn+1-mj} V_m^j, \tag{26}$$

$$V_{mn} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} V_{mn-mj} V_m^j. \tag{27}$$

Proof. Let $M = \begin{pmatrix} k & \alpha \\ -\beta & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & -\alpha \\ \beta & k \end{pmatrix}$. From Corollary 2, we obtain

$$\begin{aligned} M^m + N^m &= \begin{pmatrix} U_{m+1} & \alpha U_m \\ -\beta U_m & tU_{m-1} \end{pmatrix} + \begin{pmatrix} tU_{m-1} & -\alpha U_m \\ \beta U_m & U_{m+1} \end{pmatrix} \\ &= \begin{pmatrix} V_m & 0 \\ 0 & V_m \end{pmatrix} = V_m I, \\ M^{mn} &= \begin{pmatrix} U_{mn+1} & \alpha U_{mn} \\ -\beta U_{mn} & tU_{mn-1} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} M^{mn} &= (M^m)^n = (-N^m + V_m I)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} (-N^m)^{n-j} (V_m)^j \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} N^{mn-mj} V_m^j. \end{aligned} \tag{28}$$

By using (28), we find that

$$\begin{pmatrix} U_{mn+1} & \alpha U_{mn} \\ -\beta U_{mn} & tU_{mn-1} \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n} tU_{mn-1-mj} w_j & -\alpha \sum_{j=0}^{n} U_{mn-mj} w_j \\ \beta \sum_{j=0}^{n} U_{mn-mj} w_j & \sum_{j=0}^{n} U_{mn+1-mj} w_j \end{pmatrix}$$

where $w_j = \binom{n}{j}(-1)^{n-j} V_m^j$. The identities (24), (25), and (26) are obtained easily. The equation (27) can be shown by using the identities (2), (25), and (26). □

Theorem 12. *The following hold true:*

If n is even natural number, then

$$\begin{aligned} (k^2+4t)^{\frac{n}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j+1} = \sum_{j=0}^{n} \binom{n}{j} t^{j+1} U_{n-2j-1}, \\ 0 &= \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j}, \\ 2(k^2+4t)^{\frac{n}{2}} &= \sum_{j=0}^{n} \binom{n}{j} t^j V_{n-2j} \end{aligned}$$

and if n is odd natural number, then

$$V_n = k\sum_{j=0}^{n-1} t^j U_{n-2j} = k\sum_{j=0}^{n-1} t^{j+1} U_{n-2j-2},$$

$$k(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j+1} = -\sum_{j=0}^{n} \binom{n}{j} t^{j+1} U_{n-2j-1},$$

$$2(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{n-2j},$$

$$0 = \sum_{j=0}^{n-1} t^j U_{n-2j-1} = \sum_{j=0}^{n} \binom{n}{j} t^j V_{n-2j}.$$

Proof. Let $N = \begin{pmatrix} 0 & \alpha \\ -\beta & k \end{pmatrix}$ and $M = \begin{pmatrix} k & -\alpha \\ \beta & 0 \end{pmatrix}$. Then

$$MN = NM = -tI, N+M = kI, M-N = \begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix}.$$

On the other hand, using Corollary 2, Theorem 4, and the identity (2), we get :

$$(M-N)^n = \begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix}^n = (k^2+4t)^{n/2} I \quad \text{for even } n, \tag{29}$$

$$(M-N)^n = \begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix}^n = (k^2+4t)^{(n-1)/2} \begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix} \text{ for odd } n, \tag{30}$$

$$N^n = \begin{pmatrix} tU_{n-1} & \alpha U_n \\ -\beta U_n & U_{n+1} \end{pmatrix},$$

$$M^n = \begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix},$$

and

$$N^n + M^n = \begin{pmatrix} tU_{n-1} & \alpha U_n \\ -\beta U_n & U_{n+1} \end{pmatrix} + \begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix} = \begin{pmatrix} V_n & 0 \\ 0 & V_n \end{pmatrix}. \tag{31}$$

Moreover, we have

$$N^n + M^n = (N+M)\sum_{j=0}^{n-1} (-1)^j M^j N^{n-1-j} = k\sum_{j=0}^{n-1} N^{n-1-2j} t^j \text{ for odd } n \tag{32}$$

and

$$(M-N)^n = \sum_{j=0}^{n} \binom{n}{n-j} M^{n-j} (-N)^j = \sum_{j=0}^{n} \binom{n}{n-j} M^{n-2j} t^j. \tag{33}$$

Thus, if n is odd natural number, then we obtain

$$\begin{pmatrix} V_n & 0 \\ 0 & V_n \end{pmatrix} = k\sum_{j=0}^{n-1} N^{n-1-2j}t^j = \begin{pmatrix} kt\sum_{j=0}^{n-1} t^jU_{n-2j-2} & \alpha k\sum_{j=0}^{n-1} t^jU_{n-1-2j} \\ -\beta k\sum_{j=0}^{n-1} t^jU_{n-1-2j} & k\sum_{j=0}^{n-1} t^jU_{n-2j} \end{pmatrix}$$

by using (31) and (32) and

$$\left(k^2+4t\right)^{\frac{n-1}{2}} \begin{pmatrix} k & -2\alpha \\ 2\beta & -k \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n}\binom{n}{j}t^jU_{n-2j+1} & -\alpha\sum_{j=0}^{n}\binom{n}{j}t^jU_{n-2j} \\ \beta\sum_{j=0}^{n}\binom{n}{j}t^jU_{n-2j} & \sum_{j=0}^{n}\binom{n}{j}t^{j+1}U_{n-2j-1} \end{pmatrix}$$

by using (30) and (33). Moreover, if n is even natural number, then we obtain

$$\begin{aligned}\left(k^2+4t\right)^{\frac{n}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \sum_{j=0}^{n}\binom{n}{n-j}M^{n-2j}t^j \\ &= \begin{pmatrix} \sum_{j=0}^{n}\binom{n}{j}t^jU_{n-2j+1} & -\alpha\sum_{j=0}^{n}\binom{n}{j}t^jU_{n-2j} \\ \beta\sum_{j=0}^{n}\binom{n}{j}t^jU_{n-2j} & \sum_{j=0}^{n}\binom{n}{j}t^{j+1}U_{n-2j-1} \end{pmatrix}\end{aligned}$$

by using (29) and (33). The above identities follows easily from the matrix equality and the identity (2). □

Theorem 13. *For any natural numbers m and n, the following hold true:*

$$0 = \sum_{j=0}^{n}\binom{n}{j}(-t)^{mj}\frac{U_{mn-2mj}}{U_m},$$

$$U_mV_m^n = \sum_{j=0}^{n}\binom{n}{j}(-t)^{mj}U_{mn-2mj+m} = -\sum_{j=0}^{n}\binom{n}{j}(-t)^{mj+m}U_{mn-2mj-m}.$$

Proof. Let $M = \begin{pmatrix} 0 & -b \\ a & a+b \end{pmatrix}$ and $N = \begin{pmatrix} a+b & b \\ -a & 0 \end{pmatrix}$. Here, substituting a with α^m and b with β^m, we have $M = \begin{pmatrix} 0 & -\beta^m \\ \alpha^m & V_m \end{pmatrix}$ and $N = \begin{pmatrix} V_m & \beta^m \\ -\alpha^m & 0 \end{pmatrix}$. Clearly,

$$M+N = \begin{pmatrix} 0 & -\beta^m \\ \alpha^m & V_m \end{pmatrix} + \begin{pmatrix} V_m & \beta^m \\ -\alpha^m & 0 \end{pmatrix} = \begin{pmatrix} V_m & 0 \\ 0 & V_m \end{pmatrix} = V_mI$$

and

$$MN = MN = (-t)^mI. \tag{34}$$

In this case,

$$V_m^nI = (V_mI)^n = (M+N)^n = \sum_{j=0}^{n}\binom{n}{j}M^{n-j}N^j. \tag{35}$$

Furthermore, by using (34), we have

$$M^{n-j}N^j = M^{n-2j}(-t)^{mj}. \tag{36}$$

Thus, by using (35) and (36), we get

$$V_m^n I = \sum_{j=0}^{n} \binom{n}{j} M^{n-2j}(-t)^{mj}.$$

By using Theorem 1, we obtain

$$M^n = \begin{pmatrix} -(-t)^m \frac{U_{mn-m}}{U_m} & -\beta^m \frac{U_{mn}}{U_m} \\ \alpha^m \frac{U_{mn}}{U_m} & \frac{U_{mn+m}}{U_m} \end{pmatrix}.$$

Then, it follows that

$$\begin{pmatrix} V_m^n & 0 \\ 0 & V_m^n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^n \binom{n}{j}(-(-t)^m)(-t)^{mj}\frac{U_{m(n-2j)-m}}{U_m} & \sum_{j=0}^n \binom{n}{j}(-\beta^m)(-t)^{mj}\frac{U_{m(n-2j)}}{U_m} \\ \sum_{j=0}^n \binom{n}{j}\alpha^m(-t)^{mj}\frac{U_{m(n-2j)}}{U_m} & \sum_{j=0}^n \binom{n}{j}(-t)^{mj}\frac{U_{m(n-2j)+m}}{U_m} \end{pmatrix}.$$

□

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