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New One-parameter Integral Formulas and Inequalities of the Logarithmic Type

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Abstract

This article deals with two fundamental topics in mathematical analysis: the formulation of integral expressions and the derivation of integral inequalities. In particular, it introduces new one-parameter integral formulas and inequalities of the logarithmic type, where the integrands involve the logarithmic function in one way or another. Among the results are weighted Hölder-type integral inequalities and two different forms of Hardy-Hilbert-type integral inequalities. These results are illustrated by various examples and accompanied by rigorous proofs.

1 Introduction

The concept of the integral is central to mathematical analysis. It is used to measure areas, volumes and other quantities resulting from continuous change. In particular, integral calculus forms the basis of many advanced fields, including Fourier analysis, operator theory and probability theory. These fields primarily use integrals to analyze and understand complex phenomena. Key tools in this area also include integral formulas and inequalities. In particular, integral formulas allow certain quantities to be evaluated directly. Conversely, integral inequalities help to bound and compare integrals under different conditions. A large collection of integral formulas can be found in [1]. Famous books on integral inequalities include [2–6].

Among the best known integral inequalities are the Hölder and Hardy-Hilbert integral inequalities. The formal statement of the Hölder integral inequality for non-negative functions is given below. Let $p > 1$, $q = p/(p-1)$ be the Hölder conjugate of p, satisfying the identity $1/p + 1/q = 1$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} f^p(x)dx < +\infty$ and $\int_0^{+\infty} g^q(y)dy < +\infty$. Then we have

$$\int_0^{+\infty} f(x)g(x)dx \leq \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

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The upper bound, including the constant factor equal to 1, is sharp. Within the same framework, the Hardy-Hilbert integral inequality states that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(x)dx \leq \frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The upper bound, including the constant factor equal to $\pi/\sin(\pi/p)$, is known to be sharp. These two inequalities have inspired a great deal of research. Often, extensions introduce additional parameters, weight functions, higher dimensions, or modified functional forms, resulting in new variants. For a detailed overview of these developments, see the survey [20] and the books [2, 8]. Recently, there has been a growing interest in integral inequalities with innovative functional structures, extending their range of applications.

In this article, we focus on Hardy-Hilbert-type integral inequalities, specifically those of the logarithmic type, also known as logarithmic-type Hardy-Hilbert integral inequalities. These inequalities have the feature of incorporating a logarithmic function in one way or another. We now present a brief overview of the subject, beginning with a key result from [2]. Let $p > 1$, $q = p/(p-1)$ and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} f^p(x)dx < +\infty$ and $\int_0^{+\infty} g^q(y)dy < +\infty$. Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\log(x/y)}{x-y}f(x)g(y)dxdy \leq \left[\frac{\pi}{\sin(\pi/p)}\right]^2\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

In a similar framework, [10] provides the following result:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{|\log(x/y)|}{\max(x,y)}f(x)g(y)dxdy \leq (p^2+q^2)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

A complementary result is given in [11]. For the special case $p = 2$, it ensures that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{|\log(x/y)|}{x+y}f(x)g(y)dxdy \leq 8\eta\left[\int_0^{+\infty} f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty} g^2(y)dy\right]^{1/2},$$

where

$$\eta = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)^2} \approx 0.915966,$$

which corresponds to the Catalan constant.

An additional related result from [12] is presented below. Let $p > 1$, $q = p/(p-1)$, $r, s, \epsilon > 0$ and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{p(1-\epsilon/r)-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q(1-\epsilon/s)-1} g^q(y)dy < +\infty$. Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\log(x/y)}{x^\epsilon - y^\epsilon}f(x)g(y)dxdy$$
$$\leq \left[\frac{\pi}{\sin(\pi/r)}\right]^2\left[\int_0^{+\infty} x^{p(1-\epsilon/r)-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q(1-\epsilon/s)-1} g^q(y)dy\right]^{1/q}.$$

We may also mention some more contemporary results in [13] for the case $p = 2$, as stated below. Let $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{-1} f^2(x)dx < +\infty$ and $\int_0^{+\infty} y^{-1} g^2(y)dy < +\infty$. Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{[\log(x/y)]^2}{x^2+xy+y^2} f(x)g(y)dxdy \\ \leq \frac{16\sqrt{3}\pi}{243}\left[\int_0^{+\infty} x^{-1}f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty} y^{-1}g^2(y)dy\right]^{1/2},$$

where $16\sqrt{3}\pi/243 \approx 0.35828$.

A slight variation yields

$$\int_0^{+\infty}\int_0^{+\infty} \frac{[\log(x/y)]^2}{x^2-xy+y^2} f(x)g(y)dxdy \\ \leq \frac{43\sqrt{3}\pi}{36}\left[\int_0^{+\infty} x^{-1}f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty} y^{-1}g^2(y)dy\right]^{1/2},$$

where $43\sqrt{3}\pi/36 \approx 6.49944$.

Another complementary result from [13] is given below. Let $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{-1} f^2(x)dx < +\infty$ and $\int_0^{+\infty} y^{-3} g^2(y)dy < +\infty$. Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(x/y)}{(x-y)(x^2+y^2)} f(x)g(y)dxdy \\ \leq \frac{3\pi^2}{16}\left[\int_0^{+\infty} x^{-1}f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty} y^{-3}g^2(y)dy\right]^{1/2},$$

where $3\pi^2/16 \approx 1.85055$.

We conclude this overview with a recent result from [14], which has the capability to deal with three adjustable parameters. Let $\sigma \geq 1$, $\mu, \omega > 0$, $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} f^2(x)dx < +\infty$ and $\int_0^{+\infty} g^2(y)dy < +\infty$. Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\sigma+\mu x/y)}{\omega x+y} f(x)g(y)dxdy \\ \leq \frac{2\pi}{\omega^{1/2}}\log\left[\sigma^{1/2}+\left(\frac{\mu}{\omega}\right)^{1/2}\right]\left[\int_0^{+\infty} f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty} g^2(y)dy\right]^{1/2}.$$

In particular, if we take $\sigma = 1$, $\mu = 1$ and $\omega = 1$, then it reduces to

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(1+x/y)}{x+y} f(x)g(y)dxdy \leq 2\pi\log(2)\left[\int_0^{+\infty} f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty} g^2(y)dy\right]^{1/2},$$

where $2\pi\log(2) \approx 4.35517$.

These inequalities are innovative because of the structure of their integrands and constant factors, which are expressed in a way that guarantees sharpness. They thus offer valuable tools for solving intricate integral problems involving logarithmic functions and sophisticated functional forms. However, further research in this area remains necessary, particularly in light of potential new developments in logarithmic-type integral formulas.

1.1 Contributions

In continuation of the above results, this article introduces new logarithmic-type integral formulas that are not included in [1]. These formulas are distinguished by their tractability and dependence on an adjustable parameter. To illustrate this, we present a particularly elegant and simple result below. Let $\gamma \in [0, 4)$. Then we have

$$\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2+(2-\gamma)t+1}\right] dt = 4 \arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right].$$

These integral formulas serve as the basis for deriving new weighted Hölder-type integral inequalities and two different forms of Hardy-Hilbert-type integral inequalities. These inequalities also depend on an adjustable parameter, allowing greater flexibility and applicability. One example is presented below. Let $p > 1$, $q = p/(p-1)$, $\gamma \in (0, 4)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{p-1} f^p(x) dx < +\infty$ and $\int_0^{+\infty} y^{q-1} g^q(y) dy < +\infty$. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \log\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right] f(x)g(y) dx dy$$
$$\leq 4 \arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \left[\int_0^{+\infty} x^{p-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-1} g^q(y) dy\right]^{1/q}.$$

In particular, the obtained inequalities obtained have a completely different form from those in [2, 10–12, 14] (see the previous subsection for details). In this way, we complete the collection of logarithmic Hardy-Hilbert-type integral inequalities by introducing a new integral formula approach that emphasizes simplicity, adaptability, and analytical power.

1.2 Organization

Section 2 presents the new integral formulas, including those of the logarithmic type. Section 3 builds on these formulas to derive new integral inequalities. Finally, Section 4 offers concluding remarks and discusses possible extensions and applications.

2 New Integral Formulas

2.1 A simple integral formula

The proposition below gives a simple formula for a special one-parameter integral. The proof is mainly based on the square root and arctangent primitives.

Proposition 2.1. *Let $\alpha \in (-2, 2]$. Then we have*

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}}dt = \begin{cases} 2 & \text{if } \alpha = 2 \\ \dfrac{4}{\sqrt{4-\alpha^2}}\left\{\dfrac{\pi}{2} - \arctan\left[\dfrac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} & \text{if } \alpha \in (-2,2) \end{cases}.$$

Proof of Proposition 2.1. For $\alpha = 2$, using a suitable expression for the denominator, and the ratio and square root primitives, we get

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}}dt = \int_0^{+\infty} \frac{t^{-1/2}}{1+t+2\sqrt{t}}dt = \int_0^{+\infty} \frac{t^{-1/2}}{[1+\sqrt{t}]^2}dt = \left[-2\frac{1}{1+\sqrt{t}}\right]_{t=0}^{t\to+\infty} = 2.$$

For $\alpha \in (-2, 2)$, arranging the expression of the denominator, and using the arctangent and square root primitives, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}}dt = \int_0^{+\infty} \frac{t^{-1/2}}{1-\alpha^2/4+(\sqrt{t}+\alpha/2)^2}dt \\
&= \frac{1}{1-\alpha^2/4}\int_0^{+\infty} \frac{t^{-1/2}}{1+\{[\sqrt{t}+\alpha/2]/\sqrt{1-\alpha^2/4}\}^2}dt \\
&= \frac{1}{1-\alpha^2/4}\left[2\sqrt{1-\frac{\alpha^2}{4}}\arctan\left[\frac{\sqrt{t}+\alpha/2}{\sqrt{1-\alpha^2/4}}\right]\right]_{t=0}^{t\to+\infty} \\
&= \frac{2}{\sqrt{1-\alpha^2/4}}\left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha/2}{\sqrt{1-\alpha^2/4}}\right]\right\} \\
&= \frac{4}{\sqrt{4-\alpha^2}}\left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\}.
\end{aligned}$$

Thus, Proposition 2.1 is proved. □

We can also prove that the integral is divergent for $\alpha < 2$ and convergent for $\alpha > 2$. However, its analytical expression in the case $\alpha > 2$ remains undetermined.

Using the arctangent identities $\arctan(a) + \arctan(1/a) = \pi/2$ if $a > 0$, and $\arctan(a) + \arctan(1/a) =$

$-\pi/2$ if $a < 0$, the integral formula in Proposition 2.1 can also be expressed as

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}}dt = \begin{cases} 2 & \text{if } \alpha = 2, \\ \dfrac{4}{\sqrt{4-\alpha^2}} \arctan\left[\dfrac{\sqrt{4-\alpha^2}}{\alpha}\right] & \text{if } \alpha \in (0,2), \\ \dfrac{4}{\sqrt{4-\alpha^2}} \left\{\pi + \arctan\left[\dfrac{\sqrt{4-\alpha^2}}{\alpha}\right]\right\} & \text{if } \alpha \in (-2,0). \end{cases}$$

As these different expressions for $\alpha \in (-2,0)$ and $\alpha \in (0,2)$ offer no practical advantage for further development, we will focus on the original formulation in Proposition 2.1.

Some special examples of integral values are given below. If we take $\alpha = 0$, using the identity $\arctan(0) = 0$, then we have

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t}dt = \pi.$$

If we take $\alpha = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t+\sqrt{t}}dt = \frac{4\pi}{3\sqrt{3}}.$$

As a last example, if we take $\alpha = -1$, using the identity $\arctan[-1/\sqrt{3}] = -\arctan[1/\sqrt{3}] = -\pi/6$, then we obtain

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t-\sqrt{t}}dt = \frac{8\pi}{3\sqrt{3}}.$$

Such simple one-parameter formulas can be used to derive more complex results. This topic is covered in the subsection below.

2.2 New one-parameter logarithmic-type integral formulas

The proposition below introduces a new one-parameter logarithmic-type integral formula involving the arctangent function. Its proof relies mainly on integrating the result in Proposition 2.1, treating the parameter α as a variable, and using the logarithmic and arctangent primitives.

Proposition 2.2. *Let $\beta \in (-2,2)$. Then we have*

$$\int_0^{+\infty} \frac{1}{t} \log\left[1+\beta\frac{\sqrt{t}}{1+t}\right] dt = 2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right].$$

Proof of Proposition 2.2. Proposition 2.1 implies that, for any $\alpha \in (-2, 2)$, we have

$$\int_0^{+\infty} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}} dt = \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right] \right\}.$$

Integrating both sides with respect to α with $\alpha \in (0, \beta)$, recalling that $\beta \in (-2, 2)$ (it can be negative), we get the identity

$$A = A_\star,$$

where

$$A = \int_0^{\beta} \left\{ \int_0^{+\infty} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}} dt \right\} d\alpha$$

and

$$A_\star = \int_0^{\beta} \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right] \right\} d\alpha.$$

For the term A, the Fubini-Tonelli integral theorem allows the order of integration to be exchanged since the integrand is non-negative. This, followed by the use of the square root and logarithmic primitive, gives

$$\begin{aligned} A &= \int_0^{+\infty} \left\{ \int_0^{\beta} \frac{t^{-1/2}}{1+t+\alpha\sqrt{t}} d\alpha \right\} dt = \int_0^{+\infty} \left[\log\left[1+t+\alpha\sqrt{t}\right] \right]_{\alpha=0}^{\alpha=\beta} dt \\ &= \int_0^{+\infty} \left\{ \log\left[1+t+\beta\sqrt{t}\right] - \log(1+t) \right\} dt = \int_0^{+\infty} \frac{1}{t} \log\left[1+\beta\frac{\sqrt{t}}{1+t}\right] dt. \end{aligned}$$

For the term $A_\star$, developing basically the integral, and using the square root and arctangent primitives, we obtain

$$\begin{aligned} A_\star &= 2\pi \int_0^{\beta} \frac{1}{\sqrt{4-\alpha^2}} d\alpha - \int_0^{\beta} \frac{4}{\sqrt{4-\alpha^2}} \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right] d\alpha \\ &= 2\pi \left[\arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right] \right]_{\alpha=0}^{\alpha=\beta} - \left[2\arctan^2\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right] \right]_{\alpha=0}^{\alpha=\beta} \\ &= 2\pi \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] - 2\arctan^2\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \\ &= 2\left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]. \end{aligned}$$

So we have

$$\int_0^{+\infty} \frac{1}{t} \log\left[1+\beta\frac{\sqrt{t}}{1+t}\right] dt = 2\left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right].$$

This concludes the proof of Proposition 2.2. □

To our knowledge, this formula does not appear in the reference book [1]. By analyzing the sign of the integrand and the resulting expression, we can also derive an absolute value version of the formula, which is stated as follows:

$$\int_0^{+\infty} \frac{1}{t} \left| \log\left[1 + \beta \frac{\sqrt{t}}{1+t}\right] \right| dt = 2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\} \arctan\left[\frac{|\beta|}{\sqrt{4-\beta^2}}\right].$$

However, we do not make use of it in the present study.

Some special examples of Proposition 2.2 are given below. If we take $\beta = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we have

$$\int_0^{+\infty} \frac{1}{t} \log\left[1 + \frac{\sqrt{t}}{1+t}\right] dt = \frac{5\pi^2}{18}.$$

If we take $\beta = -1$, using the identity $\arctan[-1/\sqrt{3}] = -\arctan[1/\sqrt{3}] = -\pi/6$, then we obtain

$$\int_0^{+\infty} \frac{1}{t} \log\left[1 - \frac{\sqrt{t}}{1+t}\right] dt = -\frac{7\pi^2}{18}.$$

The proposition below presents a condensed integral formula, derived as a consequence of Proposition 2.2.

Proposition 2.3. *Let $\beta \in (0,2)$. Then we have*

$$\int_0^{+\infty} \frac{1}{t} \log\left[1 - \beta^2 \frac{t}{(1+t)^2}\right] dt = -4\arctan^2\left[\frac{\beta}{\sqrt{4-\beta^2}}\right].$$

Proof of Proposition 2.3. Using a basic logarithmic property, Proposition 2.2 two times and the fact that the arctangent function is odd, we have

$$\int_0^{+\infty} \frac{1}{t} \log \left[1 - \beta^2 \frac{t}{(1+t)^2}\right] dt = \int_0^{+\infty} \frac{1}{t} \log \left\{ \left[1 - \beta \frac{\sqrt{t}}{1+t}\right] \left[1 + \beta \frac{\sqrt{t}}{1+t}\right] \right\} dt$$

$$= \int_0^{+\infty} \frac{1}{t} \log \left[1 - \beta \frac{\sqrt{t}}{1+t}\right] dt + \int_0^{+\infty} \frac{1}{t} \log \left[1 + \beta \frac{\sqrt{t}}{1+t}\right] dt$$

$$= 2 \left\{ \pi - \arctan \left[\frac{-\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan \left[\frac{-\beta}{\sqrt{4-\beta^2}}\right]$$

$$+ 2 \left\{ \pi - \arctan \left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan \left[\frac{\beta}{\sqrt{4-\beta^2}}\right]$$

$$= -2 \left\{ \pi + \arctan \left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan \left[\frac{\beta}{\sqrt{4-\beta^2}}\right]$$

$$+ 2 \left\{ \pi - \arctan \left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan \left[\frac{\beta}{\sqrt{4-\beta^2}}\right]$$

$$= -4 \arctan^2 \left[\frac{\beta}{\sqrt{4-\beta^2}}\right].$$

This concludes the proof of Proposition 2.3. □

This formula is also not contained in [1] and is original in form. Note that all of the terms on both sides are negative.

We conclude this section with an elegant variant of Proposition 2.3. This result incorporates a more comprehensive rational polynomial expression within the main logarithmic term of the integral, and deals with positive terms on both sides.

Proposition 2.4. *Let $\gamma \in [0, 4)$. Then we have*

$$\int_0^{+\infty} \frac{1}{t} \log \left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] dt = 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}}\right].$$

Proof of Proposition 2.4. It follows from Proposition 2.3 applied with $\beta = \sqrt{\gamma}$ that

$$\int_0^{+\infty} \frac{1}{t} \log \left[1 - \gamma \frac{t}{(1+t)^2}\right] dt = -4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}}\right].$$

This, together with the management of the minus sign and a basic property of the logarithmic function, gives

$$\int_0^{+\infty} \frac{1}{t} \log \left\{ \left[1 - \gamma \frac{t}{(1+t)^2}\right]^{-1} \right\} dt = 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}}\right].$$

Developing the term into the logarithmic function yields

$$\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2+(2-\gamma)t+1}\right] dt = 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right].$$

This concludes the proof of Proposition 2.4. □

In particular, if we take $\gamma = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2+t+1}\right] dt = \frac{\pi^2}{9}.$$

If we take $\gamma = 2$, using the identity $\arctan(1) = \pi/4$, then we obtain the following simple and elegant formula:

$$\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2+1}\right] dt = \frac{\pi^2}{4}.$$

If we take $\gamma = 3$, using the identity $\arctan[\sqrt{3}] = \pi/3$, then we find that

$$\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2-t+1}\right] dt = \frac{4\pi^2}{9}.$$

These formulas have a wide range of potential applications. These include evaluating complex integrals, developing analytical bounds, and refining mathematical models involving logarithmic terms. In this study, we use them to make new contributions to the field of integral inequalities, as discussed in the section below.

3 Integral Inequalities

This section is mainly devoted to new weighted Hölder-type integral inequalities and new Hardy-Hilbert integral inequalities, with a focus on the logarithmic type.

3.1 New weighted Hölder-type integral inequalities

The proposition below introduces the topic by presenting a simple weighted Hölder-type integral inequality, which is derived from Proposition 2.1. Although the inequality is not of logarithmic type, it is of interest from an analytical point of view.

Proposition 3.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha \in (-2, 2)$ and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that $\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt < +\infty$. Then we have*

$$\int_0^{+\infty} \frac{1}{\left[1 + t + \alpha\sqrt{t}\right]^{1/p}} f(t)dt \leq \frac{4^{1/p}}{(4-\alpha^2)^{1/(2p)}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\}^{1/p} \left[\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt\right]^{1/q}.$$

Proof of Proposition 3.1. By means of a suitable product decomposition of the integrand with the aim of using Proposition 2.1, followed by the Hölder integral inequality, we find that

$$\int_0^{+\infty} \frac{1}{\left[1 + t + \alpha\sqrt{t}\right]^{1/p}} f(t)dt = \int_0^{+\infty} \frac{t^{-1/(2p)}}{\left[1 + t + \alpha\sqrt{t}\right]^{1/p}} t^{1/(2p)} f(t)dt \leq \left[\int_0^{+\infty} \frac{t^{-1/2}}{1 + t + \alpha\sqrt{t}} dt\right]^{1/p} \left[\int_0^{+\infty} t^{q/(2p)} f^q(t)dt\right]^{1/q}.$$

By virtue of Proposition 2.1 with $\alpha \in (-2, 2)$ and the identity $q/(2p) = (q-1)/2$, we obtain

$$\left[\int_0^{+\infty} \frac{t^{-1/2}}{1 + t + \alpha\sqrt{t}} dt\right]^{1/p} \left[\int_0^{+\infty} t^{q/(2p)} f^q(t)dt\right]^{1/q} = \frac{4^{1/p}}{(4-\alpha^2)^{1/(2p)}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\}^{1/p} \left[\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt\right]^{1/q}.$$

So we have

$$\int_0^{+\infty} \frac{1}{\left[1 + t + \alpha\sqrt{t}\right]^{1/p}} f(t)dt \leq \frac{4^{1/p}}{(4-\alpha^2)^{1/(2p)}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\}^{1/p} \left[\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt\right]^{1/q}.$$

This concludes the proof of Proposition 3.1. □

In particular, if we take $\alpha = 0$, then this inequality reduces to

$$\int_0^{+\infty} \frac{1}{(1+t)^{1/p}} f(t)dt \leq \pi^{1/p} \left[\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt\right]^{1/q}.$$

If we take $\alpha = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we have

$$\int_0^{+\infty} \frac{1}{\left[1 + t + \sqrt{t}\right]^{1/p}} f(t)dt \leq \frac{4^{1/p}\pi^{1/p}}{3^{3/(2p)}} \left[\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt\right]^{1/q}.$$

If we take $\alpha = -1$, using the identity $\arctan[-1/\sqrt{3}] = -\arctan[1/\sqrt{3}] = -\pi/6$, then we obtain

$$\int_0^{+\infty} \frac{1}{\left[1+t-\sqrt{t}\right]^{1/p}} f(t)dt \leq \frac{8^{1/p}\pi^{1/p}}{3^{3/(2p)}} \left[\int_0^{+\infty} t^{(q-1)/2} f^q(t)dt\right]^{1/q}.$$

Our first weighted logarithmic Hölder-type integral inequality is presented below. It is based mainly on Proposition 2.2, focusing on the range of values $\beta \in [0, 2)$, for which the integral expression remains positive.

Proposition 3.2. *Let $p > 1$, $q = p/(p-1)$, $\beta \in [0, 2)$ and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that $\int_0^{+\infty} t^{q-1} f^q(t)dt < +\infty$. Then we have*

$$\begin{aligned}
&\int_0^{+\infty} \log^{1/p}\left[1+\beta\frac{\sqrt{t}}{1+t}\right] f(t)dt \\
&\leq 2^{1/p}\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}^{1/p} \arctan^{1/p}\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.
\end{aligned}$$

Proof of Proposition 3.2. By means of a suitable product decomposition of the integrand with the aim of using Proposition 2.2, followed by the Hölder integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty} \log^{1/p}\left[1+\beta\frac{\sqrt{t}}{1+t}\right] f(t)dt = \int_0^{+\infty} \frac{1}{t^{1/p}} \log^{1/p}\left[1+\beta\frac{\sqrt{t}}{1+t}\right] t^{1/p} f(t)dt \\
&\leq \left[\int_0^{+\infty} \frac{1}{t} \log\left[1+\beta\frac{\sqrt{t}}{1+t}\right] dt\right]^{1/p} \left[\int_0^{+\infty} t^{q/p} f^q(t)dt\right]^{1/q}.
\end{aligned}$$

By virtue of Proposition 2.2 with $\beta \in [0, 2)$ and the identity $q/p = q - 1$, we obtain

$$\begin{aligned}
&\left[\int_0^{+\infty} \frac{1}{t} \log^{1/p}\left[1+\beta\frac{\sqrt{t}}{1+t}\right] dt\right]^{1/p} \left[\int_0^{+\infty} t^{q/p} f^q(t)dt\right]^{1/q} \\
&= 2^{1/p}\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}^{1/p} \arctan^{1/p}\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.
\end{aligned}$$

So we have

$$\begin{aligned}
&\int_0^{+\infty} \log^{1/p}\left[1+\beta\frac{\sqrt{t}}{1+t}\right] f(t)dt \\
&\leq 2^{1/p}\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}^{1/p} \arctan^{1/p}\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.
\end{aligned}$$

This concludes the proof of Proposition 3.2. □

As an example, if we take $\beta = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we have

$$\int_0^{+\infty} \log^{1/p}\left[1 + \frac{\sqrt{t}}{1+t}\right] f(t)dt \leq \frac{5^{1/p}\pi^{2/p}}{18^{1/p}} \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

Another weighted logarithmic Hölder-type integral inequality is explored below. It uses mainly Proposition 2.4.

Proposition 3.3. *Let $p > 1$, $q = p/(p-1)$, $\gamma \in [0, 4)$ and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that $\int_0^{+\infty} t^{q-1} f^q(t)dt < +\infty$. Then we have*

$$\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] f(t)dt \leq 4^{1/p} \arctan^{2/p}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

Proof of Proposition 3.3. By means of a suitable product decomposition of the integrand with the aim of using Proposition 2.4, followed by the Hölder integral inequality, we get

$$\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] f(t)dt = \int_0^{+\infty} \frac{1}{t^{1/p}} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] t^{1/p} f(t)dt$$
$$\leq \left\{\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] dt\right\}^{1/p} \left[\int_0^{+\infty} t^{q/p} f^q(t)dt\right]^{1/q}.$$

By virtue of Proposition 2.4 and the identity $q/p = q - 1$, we obtain

$$\left\{\int_0^{+\infty} \frac{1}{t} \log\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] dt\right\}^{1/p} \left[\int_0^{+\infty} t^{q/p} f^q(t)dt\right]^{1/q}$$
$$= 4^{1/p} \arctan^{2/p}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q} \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

So we have

$$\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] f(t)dt \leq 4^{1/p} \arctan^{2/p}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

Thus, Proposition 3.3 is proved. □

In particular, if we take $\gamma = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + t + 1}\right] f(t)dt \leq \frac{\pi^{2/p}}{9^{1/p}} \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

If we take $\gamma = 2$, using the identity $\arctan(1) = \pi/4$, then we obtain the following elegant inequality:

$$\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + 1}\right] f(t)dt \leq \frac{\pi^{2/p}}{4^{1/p}} \left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

If we take $\gamma = 3$, using the identity $\arctan[\sqrt{3}] = \pi/3$, then we obtain

$$\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 - t + 1}\right] f(t)dt \leq \frac{4^{1/p}\pi^{2/p}}{9^{1/p}}\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}.$$

Based on these results, we can think of using a mixed inequality approach. For example, in the proposition below, we propose a mixed version of Proposition 3.3 dealing with two functions.

Proposition 3.4. *Let $p > 1$, $q = p/(p-1)$, $\gamma, \zeta \in [0, 4)$ and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} t^{q-1} f^q(t)dt < +\infty$ and $\int_0^{+\infty} t^{p-1} g^p(t)dt < +\infty$. Then we have*

$$\begin{aligned}
&\int_0^{+\infty} \log^{1/p^2}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] \log^{1/q^2}\left[\frac{(t+1)^2}{t^2 + (2-\zeta)t + 1}\right] f^{1/p}(t) g^{1/q}(t)dt \\
&\leq 4^{1/p^2 + 1/q^2} \arctan^{2/p^2}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \arctan^{2/q^2}\left[\sqrt{\frac{\zeta}{4-\zeta}}\right] \times \\
&\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/(pq)} \left[\int_0^{+\infty} t^{p-1} g^p(t)dt\right]^{1/(pq)}.
\end{aligned}$$

Proof of Proposition 3.4. By means of a suitable product decomposition of the integrand with the aim of using Proposition 3.3, followed by the Hölder integral inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \log^{1/p^2}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] \log^{1/q^2}\left[\frac{(t+1)^2}{t^2 + (2-\zeta)t + 1}\right] f^{1/p}(t) g^{1/q}(t)dt \\
&= \int_0^{+\infty} \left\{\log^{1/p}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] f(t)\right\}^{1/p} \times \\
&\left\{\log^{1/q}\left[\frac{(t+1)^2}{t^2 + (2-\zeta)t + 1}\right] g(t)\right\}^{1/q} dt \\
&\leq \left\{\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2 + (2-\gamma)t + 1}\right] f(t)dt\right\}^{1/p} \times \\
&\left\{\int_0^{+\infty} \log^{1/q}\left[\frac{(t+1)^2}{t^2 + (2-\zeta)t + 1}\right] g(t)dt\right\}^{1/q}.
\end{aligned}$$

Applying Proposition 3.3 independently to the functions f and g, and using the identities $q = p/(p-1)$

and $p = q/(q-1)$, we obtain

$$\left\{\int_0^{+\infty} \log^{1/p}\left[\frac{(t+1)^2}{t^2+(2-\gamma)t+1}\right] f(t)dt\right\}^{1/p} \times$$
$$\left\{\int_0^{+\infty} \log^{1/q}\left[\frac{(t+1)^2}{t^2+(2-\zeta)t+1}\right] g(t)dt\right\}^{1/q}$$
$$\leq \left\{4^{1/p} \arctan^{2/p}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/q}\right\}^{1/p} \times$$
$$\left\{4^{1/q} \arctan^{2/q}\left[\sqrt{\frac{\zeta}{4-\zeta}}\right]\left[\int_0^{+\infty} t^{p-1} g^p(t)dt\right]^{1/p}\right\}^{1/q}$$
$$= 4^{1/p^2+1/q^2} \arctan^{2/p^2}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \arctan^{2/q^2}\left[\sqrt{\frac{\zeta}{4-\zeta}}\right] \times$$
$$\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/(pq)}\left[\int_0^{+\infty} t^{p-1} g^p(t)dt\right]^{1/(pq)}.$$

So we have

$$\int_0^{+\infty} \log^{1/p^2}\left[\frac{(t+1)^2}{t^2+(2-\gamma)t+1}\right] \log^{1/q^2}\left[\frac{(t+1)^2}{t^2+(2-\zeta)t+1}\right] f^{1/p}(t) g^{1/q}(t)dt$$
$$\leq 4^{1/p^2+1/q^2} \arctan^{2/p^2}\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \arctan^{2/q^2}\left[\sqrt{\frac{\zeta}{4-\zeta}}\right] \times$$
$$\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/(pq)}\left[\int_0^{+\infty} t^{p-1} g^p(t)dt\right]^{1/(pq)}.$$

Thus, the proof of Proposition 3.4 is concluded. □

As an example, if we take $\gamma = 1$ and $\zeta = 3$, using the identities $\arctan[1/\sqrt{3}] = \pi/6$ and $\arctan[\sqrt{3}] = \pi/3$, then we get

$$\int_0^{+\infty} \log^{1/p^2}\left[\frac{(t+1)^2}{t^2+t+1}\right] \log^{1/q^2}\left[\frac{(t+1)^2}{t^2-t+1}\right] f^{1/p}(t) g^{1/q}(t)dt$$
$$\leq \frac{\pi^{2/p^2+2/q^2} 4^{1/q^2}}{9^{1/p^2+1/q^2}}\left[\int_0^{+\infty} t^{q-1} f^q(t)dt\right]^{1/(pq)}\left[\int_0^{+\infty} t^{p-1} g^p(t)dt\right]^{1/(pq)}.$$

The mixed approach used in Proposition 3.4 is elaborated upon by applying Proposition 3.3 to two different parameters. With minimal effort, this approach can be adapted to the other propositions in this section with different parameters or weight functions, leading to new mixed integral inequalities. While we will not develop this further here, it is clear that enriching the collection of weighted Hölder-type integral inequalities is of interest.

Continuing in the spirit of [2, 10–12, 14], the rest of the article is devoted to new Hardy-Hilbert-type integral inequalities. Two forms are distinguished: the first is characterized by the omnipresence of xy in the integral, while the second is characterized by the presence of $x+y$.

3.2 New Hardy-Hilbert-type integral inequalities of the first form

The proposition below presents a variant of the Hardy-Hilbert-type integral inequality of the first form based on Proposition 2.1.

Proposition 3.5. *Let $p>1$, $q=p/(p-1)$, $\alpha \in (-2,2)$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions such that $\int_0^{+\infty} x^{p/2-1}f^p(x)dx<+\infty$ and $\int_0^{+\infty} y^{q/2-1}g^q(y)dy<+\infty$. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+xy+\alpha\sqrt{xy}}f(x)g(y)dxdy$$
$$\leq \frac{4}{\sqrt{4-\alpha^2}}\left\{\frac{\pi}{2}-\arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.5. By appropriately decomposing the integrand using the identity $1/p+1/q=1$, and with the aim of applying Proposition 2.1 along with the Hölder integral inequality, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+xy+\alpha\sqrt{xy}}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/(2q)}y^{-1/(2p)}}{\left[1+xy+\alpha\sqrt{xy}\right]^{1/p}}f(x)\times\frac{x^{-1/(2q)}y^{1/(2p)}}{\left[1+xy+\alpha\sqrt{xy}\right]^{1/q}}g(y)dxdy\\&\leq B^{1/p}C^{1/q},\end{aligned}\tag{1}$$

where

$$B=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{p/(2q)}y^{-1/2}}{1+xy+\alpha\sqrt{xy}}f^p(x)dxdy$$

and

$$C=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{-1/2}y^{q/(2p)}}{1+xy+\alpha\sqrt{xy}}g^q(y)dxdy.$$

Let us examine B and C, one after the other.

Since the integrand associated with B is non-negative, we can apply the Fubini-Tonelli integral theorem, ensuring the exchange of the order of integration. Then, performing the change of variables $u=xy$ with respect to y, using the identity $p/(2q) = (p-1)/2$, and applying Proposition 2.1, we obtain

$$\begin{aligned} B &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[\int_0^{+\infty} \frac{y^{-1/2}}{1+xy+\alpha\sqrt{xy}} dy\right] dx \\ &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[\int_0^{+\infty} \frac{(xy)^{-1/2}}{1+xy+\alpha\sqrt{xy}} x dy\right] dx \\ &= \int_0^{+\infty} x^{p/2-1} f^p(x) \left[\int_0^{+\infty} \frac{u^{-1/2}}{1+u+\alpha\sqrt{u}} du\right] dx \\ &= \int_0^{+\infty} x^{p/2-1} f^p(x) \times \frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} dx \\ &= \frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \end{aligned} \tag{2}$$

For the term C, we proceed similarly, using the change of variables $v = xy$ with respect to x, as follows:

$$\begin{aligned} C &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} \frac{x^{-1/2}}{1+xy+\alpha\sqrt{xy}} dx\right] dy \\ &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \frac{(xy)^{-1/2}}{1+xy+\alpha\sqrt{xy}} y dx\right] dy \\ &= \int_0^{+\infty} y^{q/2-1} g^q(y) \left[\int_0^{+\infty} \frac{v^{-1/2}}{1+v+\alpha\sqrt{v}} dv\right] dy \\ &= \int_0^{+\infty} y^{q/2-1} g^q(y) \times \frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} dy \\ &= \frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \end{aligned} \tag{3}$$

Using Equations (1), (2), and (3), together with the identity $1/p + 1/q = 1$, we obtain

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \frac{1}{1+xy+\alpha\sqrt{xy}} f(x)g(y) dx dy \\ &\leq \left[\frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} \int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \times \\ &\left[\frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} \int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q} \\ &= \frac{4}{\sqrt{4-\alpha^2}} \left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\} \times \\ &\left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}. \end{aligned}$$

This ends the proof of Proposition 3.5. □

In particular, if we take $\alpha = 0$, then we directly have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+xy}f(x)g(y)dxdy \leq \pi\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

This is a well-known product variant of the Hardy-Hilbert integral inequality, with π as the optimal constant factor.

More interestingly, if we take $\alpha = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+xy+\sqrt{xy}}f(x)g(y)dxdy$$
$$\leq \frac{4\pi}{3\sqrt{3}}\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where $4\pi/[3\sqrt{3}] \approx 2.418399$.

The proposition below presents our first logarithmic Hardy-Hilbert-type integral inequality of the first form. It is mainly based on Proposition 2.2.

Proposition 3.6. *Let $p > 1$, $q = p/(p-1)$, $\beta \in (0,2)$, and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions such that $\int_0^{+\infty} x^{p-1}f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q-1}g^q(y)dy < +\infty$. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\log\left[1+\beta\frac{\sqrt{xy}}{1+xy}\right]f(x)g(y)dxdy$$
$$\leq 2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\times$$
$$\left[\int_0^{+\infty}x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q-1}g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.6. By decomposing the integrand suitably using the identity $1/p + 1/q = 1$, and applying Proposition 2.2 along with the Hölder integral inequality, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\log\left[1+\beta\frac{\sqrt{xy}}{1+xy}\right]f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty}x^{1/q}y^{-1/p}\log^{1/p}\left[1+\beta\frac{\sqrt{xy}}{1+xy}\right]f(x)$$
$$\times\, x^{-1/q}y^{1/p}\log^{1/q}\left[1+\beta\frac{\sqrt{xy}}{1+xy}\right]g(y)dxdy$$
$$\leq D^{1/p}E^{1/q}, \tag{4}$$

where

$$D = \int_0^{+\infty} \int_0^{+\infty} x^{p/q} y^{-1} \log \left[1 + \beta \frac{\sqrt{xy}}{1+xy}\right] f^p(x) dx dy$$

and

$$E = \int_0^{+\infty} \int_0^{+\infty} x^{-1} y^{q/p} \log \left[1 + \beta \frac{\sqrt{xy}}{1+xy}\right] g^q(y) dx dy.$$

Let us study D and E, one after the other.

Since the integrand associated with D is non-negative, we can apply the Fubini-Tonelli integral theorem, ensuring the exchange of the order of integration. This, followed by the change of variables $u = xy$ with respect to y, the identity $p/q = p - 1$ and the application of Proposition 2.2, gives

$$\begin{aligned} D &= \int_0^{+\infty} x^{p/q} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{y} \log \left[1 + \beta \frac{\sqrt{xy}}{1+xy}\right] dy \right\} dx \\ &= \int_0^{+\infty} x^{p/q} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{xy} \log \left[1 + \beta \frac{\sqrt{xy}}{1+xy}\right] x dy \right\} dx \\ &= \int_0^{+\infty} x^{p-1} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{u} \log \left[1 + \beta \frac{\sqrt{u}}{1+u}\right] du \right\} dx \\ &= \int_0^{+\infty} x^{p-1} f^p(x) \times 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] dx \\ &= 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \int_0^{+\infty} x^{p-1} f^p(x) dx. \end{aligned} \tag{5}$$

For the term E, we proceed in a similar way but with the change of variables $v = xy$ with respect to x, as follows:

$$\begin{aligned} E &= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{x} \log \left[1 + \beta \frac{\sqrt{xy}}{1+xy}\right] dx \right\} dy \\ &= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{xy} \log \left[1 + \beta \frac{\sqrt{xy}}{1+xy}\right] y dx \right\} dy \\ &= \int_0^{+\infty} y^{q-1} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{v} \log \left[1 + \beta \frac{\sqrt{v}}{1+v}\right] dv \right\} dy \\ &= \int_0^{+\infty} y^{q-1} g^q(y) \times 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] dy \\ &= 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \int_0^{+\infty} y^{q-1} g^q(y) dy. \end{aligned} \tag{6}$$

By virtue of Equations (4), (5) and (6), and the identity $1/p + 1/q = 1$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[1+\beta\frac{\sqrt{xy}}{1+xy}\right] f(x)g(y)dxdy$$
$$\leq \left[2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\times$$
$$\left[2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q}$$
$$= 2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\times$$
$$\left[\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 3.6. □

As a special example, if we take $\beta = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[1+\frac{\sqrt{xy}}{1+xy}\right] f(x)g(y)dxdy$$
$$\leq \frac{5\pi^2}{18}\left[\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q},$$

where $5\pi^2/18 \approx 2.741556$.

The proposition below introduces another logarithmic Hardy-Hilbert-type integral inequality of the first form. It is derived mainly from Proposition 2.4.

Proposition 3.7. *Let $p > 1$, $q = p/(p-1)$, $\gamma \in (0,4)$, and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions such that $\int_0^{+\infty} x^{p-1}f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q-1}g^q(y)dy < +\infty$. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] f(x)g(y)dxdy$$
$$\leq 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]\left[\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.7. Using the identity $1/p + 1/q = 1$ to decompose the integrand, and applying

Proposition 2.4 along with the Hölder integral inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/q}y^{-1/p}\log^{1/p}\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] f(x) \\
&\times x^{-1/q}y^{1/p}\log^{1/q}\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] g(y)dxdy \\
&\leq F^{1/p}G^{1/q}, \qquad (7)
\end{aligned}$$

where

$$F = \int_0^{+\infty}\int_0^{+\infty} x^{p/q}y^{-1}\log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] f^p(x)dxdy$$

and

$$G = \int_0^{+\infty}\int_0^{+\infty} x^{-1}y^{q/p}\log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] g^q(y)dxdy.$$

Let us examine F and G, one after the other.

Since the integrand associated with F is non-negative, we can apply the Fubini-Tonelli integral theorem, which ensures the exchange of the order of integration. Following this, by performing the change of variables $u = xy$ with respect to y, using the identity $p/q = p-1$, and applying Proposition 2.4, we get

$$\begin{aligned}
F &= \int_0^{+\infty} x^{p/q}f^p(x)\left\{\int_0^{+\infty}\frac{1}{y}\log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right]dy\right\}dx \\
&= \int_0^{+\infty} x^{p/q}f^p(x)\left\{\int_0^{+\infty}\frac{1}{xy}\log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right]xdy\right\}dx \\
&= \int_0^{+\infty} x^{p-1}f^p(x)\left\{\int_0^{+\infty}\frac{1}{u}\log\left[\frac{(u+1)^2}{u^2+(2-\gamma)u+1}\right]du\right\}dx \\
&= \int_0^{+\infty} x^{p-1}f^p(x)\times 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]dx \\
&= 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]\int_0^{+\infty} x^{p-1}f^p(x)dx. \qquad (8)
\end{aligned}$$

For the term G, we proceed in a similar manner, but with the change of variables $v = xy$ with respect to x, as follows:

$$
\begin{aligned}
G &= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{x} \log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] dx \right\} dy \\
&= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{xy} \log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] ydx \right\} dy \\
&= \int_0^{+\infty} y^{q-1} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{v} \log\left[\frac{(v+1)^2}{v^2+(2-\gamma)v+1}\right] dv \right\} dy \\
&= \int_0^{+\infty} y^{q-1} g^q(y) \times 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] dy \\
&= 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \int_0^{+\infty} y^{q-1} g^q(y) dy. \qquad (9)
\end{aligned}
$$

Based on Equations (7), (8) and (9), and the identity $1/p + 1/q = 1$, we obtain

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \log\left[\frac{(xy+1)^2}{x^2y^2+(2-\gamma)xy+1}\right] f(x)g(y)dxdy \\
&\leq \left\{ 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \int_0^{+\infty} x^{p-1} f^p(x) dx \right\}^{1/p} \times \\
&\left\{ 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \int_0^{+\infty} y^{q-1} g^q(y) dy \right\}^{1/q} \\
&= 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right] \left[\int_0^{+\infty} x^{p-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-1} g^q(y) dy\right]^{1/q}.
\end{aligned}
$$

Thus, Proposition 3.7 is proved. □

As a special example, if we take $\gamma = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \log\left[\frac{(xy+1)^2}{x^2y^2+xy+1}\right] f(x)g(y)dxdy \\
&\leq \frac{\pi^2}{9} \left[\int_0^{+\infty} x^{p-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-1} g^q(y) dy\right]^{1/q},
\end{aligned}
$$

where $\pi^2/9 \approx 1.09662$.

We highlight the elegant simplicity and originality of this logarithmic-type integral inequality, which features a constant factor dependent on π.

The rest of the article is devoted to new Hardy-Hilbert-type integral inequalities of the second form, i.e., those dealing with $x + y$ rather than primarily xy. These forms of inequality are more closely related to the original Hardy-Hilbert integral inequality.

3.3 New Hardy-Hilbert-type integral inequalities of the second form

The proposition below presents our first Hardy-Hilbert-type integral inequality of the second form, with Proposition 2.1 forming the core of the proof.

Proposition 3.8. *Let $p > 1$, $q = p/(p-1)$, $\alpha \in (-2, 2)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y+\alpha\sqrt{xy}} f(x)g(y)dxdy$$
$$\leq \frac{4}{\sqrt{4-\alpha^2}}\left\{\frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\right\}\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.8. By decomposing the integrand suitably using the identity $1/p + 1/q = 1$ and applying Proposition 2.1 along with the Hölder integral inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y+\alpha\sqrt{xy}} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/(2q)}y^{-1/(2p)}}{\left[x+y+\alpha\sqrt{xy}\right]^{1/p}} f(x) \times \frac{x^{-1/(2q)}y^{1/(2p)}}{\left[x+y+\alpha\sqrt{xy}\right]^{1/q}} g(y)dxdy \\
&\leq H^{1/p} I^{1/q},
\end{aligned} \tag{10}$$

where

$$H = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{p/(2q)}y^{-1/2}}{x+y+\alpha\sqrt{xy}} f^p(x)dxdy$$

and

$$I = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{-1/2}y^{q/(2p)}}{x+y+\alpha\sqrt{xy}} g^q(y)dxdy.$$

Let us study H and I, one after the other.

Since the integrand associated with H is non-negative, we can apply the Fubini-Tonelli integral theorem, which justifies the exchange of the order of integration. Then, performing the change of variables $u = y/x$

with respect to y, using the identity $p/(2q) = (p-1)/2$, and applying Proposition 2.1, we obtain

$$\begin{aligned} H &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[\int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha\sqrt{xy}} dy \right] dx \\ &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[\int_0^{+\infty} \frac{(y/x)^{-1/2}}{1+(y/x)+\alpha\sqrt{y/x}} \times \frac{1}{x} dy \right] dx \\ &= \int_0^{+\infty} x^{p/2-1} f^p(x) \left[\int_0^{+\infty} \frac{u^{-1/2}}{1+u+\alpha\sqrt{u}} du \right] dx \\ &= \int_0^{+\infty} x^{p/2-1} f^p(x) \times \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} dx \\ &= \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \end{aligned} \tag{11}$$

For the term I, we proceed in a similar manner, making the change of variables $v = x/y$ with respect to x, as follows:

$$\begin{aligned} I &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} \frac{x^{-1/2}}{x+y+\alpha\sqrt{xy}} dx \right] dy \\ &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \frac{(x/y)^{-1/2}}{x/y+1+\alpha\sqrt{x/y}} \times \frac{1}{y} dx \right] dy \\ &= \int_0^{+\infty} y^{q/2-1} g^q(y) \left[\int_0^{+\infty} \frac{v^{-1/2}}{1+v+\alpha\sqrt{v}} dv \right] dy \\ &= \int_0^{+\infty} y^{q/2-1} g^q(y) \times \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} dy \\ &= \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \end{aligned} \tag{12}$$

Based on Equations (10), (11) and (12), and the identity $1/p + 1/q = 1$, we obtain

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y+\alpha\sqrt{xy}} f(x) g(y) dx dy \\ & \leq \left[\frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} \int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \times \\ & \left[\frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} \int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q} \\ & = \frac{4}{\sqrt{4-\alpha^2}} \left\{ \frac{\pi}{2} - \arctan\left[\frac{\alpha}{\sqrt{4-\alpha^2}} \right] \right\} \times \\ & \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q}. \end{aligned}$$

This concludes the proof of Proposition 3.8. □

In particular, if we take $\alpha = 0$, then we directly have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \le \pi \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

This is a well-known variant of the Hardy-Hilbert integral inequality, with π as the optimal constant factor.

More interestingly, if we take $\alpha = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y+\sqrt{xy}} f(x)g(y)dxdy$$
$$\le \frac{4\pi}{3\sqrt{3}} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q},$$

where $4\pi/[3\sqrt{3}] \approx 2.418399$.

Thus, Proposition 3.8 extends the scope of the Hardy-Hilbert integral inequality through the introduction of the parameter α.

The proposition below presents our first logarithmic Hardy-Hilbert-type integral inequality of the second form. The proof relies mainly on Proposition 2.2.

Proposition 3.9. *Let $p > 1$, $q = p/(p-1)$, $\beta \in (0, 2)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{p-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q-1} g^q(y)dy < +\infty$. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[1+\beta\frac{\sqrt{xy}}{x+y}\right] f(x)g(y)dxdy$$
$$\le 2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \times$$
$$\left[\int_0^{+\infty} x^{p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-1} g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.9. Using the identity $1/p + 1/q = 1$ to decompose suitably the integrand, and with the aim of applying Proposition 2.2 together with the Hölder integral inequality, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[1+\beta\frac{\sqrt{xy}}{x+y}\right] f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/q}y^{-1/p} \log^{1/p}\left[1+\beta\frac{\sqrt{xy}}{x+y}\right] f(x)$$
$$\times x^{-1/q}y^{1/p} \log^{1/q}\left[1+\beta\frac{\sqrt{xy}}{x+y}\right] g(y)dxdy$$
$$\le J^{1/p}K^{1/q}, \tag{13}$$

where

$$J = \int_0^{+\infty} \int_0^{+\infty} x^{p/q} y^{-1} \log\left[1 + \beta \frac{\sqrt{xy}}{x+y}\right] f^p(x) dx dy$$

and

$$K = \int_0^{+\infty} \int_0^{+\infty} x^{-1} y^{q/p} \log\left[1 + \beta \frac{\sqrt{xy}}{x+y}\right] g^q(y) dx dy.$$

Let us examine J and K, one after the other.

Since the integrand associated with J is non-negative, we can apply the Fubini-Tonelli integral theorem to justify the exchange of the order of integration. Then, performing the change of variables $u = y/x$ with respect to y, using the identity $p/q = p - 1$, and applying Proposition 2.2, we get

$$\begin{aligned} J &= \int_0^{+\infty} x^{p/q} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{y} \log\left[1 + \beta \frac{\sqrt{xy}}{x+y}\right] dy \right\} dx \\ &= \int_0^{+\infty} x^{p/q} f^p(x) \left\{ \int_0^{+\infty} \frac{x}{y} \log\left[1 + \beta \frac{\sqrt{(y/x)}}{y/x+1}\right] \frac{1}{x} dy \right\} dx \\ &= \int_0^{+\infty} x^{p-1} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{u} \log\left[1 + \beta \frac{\sqrt{u}}{1+u}\right] du \right\} dx \\ &= \int_0^{+\infty} x^{p-1} f^p(x) \times 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] dx \\ &= 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \int_0^{+\infty} x^{p-1} f^p(x) dx. \end{aligned} \tag{14}$$

For the term K, we proceed in a similar manner, considering the change of variables $v = x/y$ with respect to x, as follows:

$$\begin{aligned} K &= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{x} \log\left[1 + \beta \frac{\sqrt{xy}}{x+y}\right] dx \right\} dy \\ &= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{y}{x} \log\left[1 + \beta \frac{\sqrt{x/y}}{x/y+1}\right] \frac{1}{y} dx \right\} dy \\ &= \int_0^{+\infty} y^{q-1} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{v} \log\left[1 + \beta \frac{\sqrt{v}}{1+v}\right] dv \right\} dy \\ &= \int_0^{+\infty} y^{q-1} g^q(y) \times 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] dy \\ &= 2 \left\{ \pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \right\} \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \int_0^{+\infty} y^{q-1} g^q(y) dy. \end{aligned} \tag{15}$$

It follows from Equations (13), (14) and (15), and the identity $1/p + 1/q = 1$, that

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[1 + \beta\frac{\sqrt{xy}}{x+y}\right] f(x)g(y)dxdy$$
$$\leq \left[2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p} \times$$
$$\left[2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q}$$
$$= 2\left\{\pi - \arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right]\right\}\arctan\left[\frac{\beta}{\sqrt{4-\beta^2}}\right] \times$$
$$\left[\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q}.$$

Thus, Proposition 3.9 is proved. □

As a special example, if we take $\beta = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[1 + \frac{\sqrt{xy}}{x+y}\right] f(x)g(y)dxdy$$
$$\leq \frac{5\pi^2}{18}\left[\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q},$$

where $5\pi^2/18 \approx 2.741556$.

The proposition below introduces another logarithmic Hardy-Hilbert-type integral inequality of the second form. It is derived mainly from Proposition 2.4.

Proposition 3.10. *Let $p > 1$, $q = p/(p-1)$, $\gamma \in (0, 4)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{p-1}f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q-1}g^q(y)dy < +\infty$. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \log\left[\frac{(x+y)^2}{x^2 + (2-\gamma)xy + y^2}\right] f(x)g(y)dxdy$$
$$\leq 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]\left[\int_0^{+\infty} x^{p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-1}g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.10. By means of a suitable product decomposition of the integrand using the identity $1/p + 1/q = 1$, and with the aim of applying Proposition 2.4 together with the Hölder integral inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\log\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right]f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}x^{1/q}y^{-1/p}\log^{1/p}\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right]f(x)\\
&\times x^{-1/q}y^{1/p}\log^{1/q}\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right]g(y)dxdy\\
&\leq L^{1/p}M^{1/q},
\end{aligned}\tag{16}$$

where

$$L=\int_0^{+\infty}\int_0^{+\infty}x^{p/q}y^{-1}\log\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right]f^p(x)dxdy$$

and

$$M=\int_0^{+\infty}\int_0^{+\infty}x^{-1}y^{q/p}\log\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right]g^q(y)dxdy.$$

Let us study L and M, one after the other.

Since the integrand associated with L is non-negative, we can apply the Fubini-Tonelli integral theorem to justify the exchange of the order of integration. This, followed by the change of variables $u=y/x$ with respect to y, the identity $p/q=p-1$, and the application of Proposition 2.4, yields

$$\begin{aligned}
L&=\int_0^{+\infty}x^{p/q}f^p(x)\left\{\int_0^{+\infty}\frac{1}{y}\log\left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2}\right]dy\right\}dx\\
&=\int_0^{+\infty}x^{p/q}f^p(x)\left\{\int_0^{+\infty}\frac{x}{y}\log\left[\frac{(1+y/x)^2}{1+(2-\gamma)(y/x)+(y/x)^2}\right]\frac{1}{x}dy\right\}dx\\
&=\int_0^{+\infty}x^{p-1}f^p(x)\left\{\int_0^{+\infty}\frac{1}{u}\log\left[\frac{(u+1)^2}{u^2+(2-\gamma)u+1}\right]du\right\}dx\\
&=\int_0^{+\infty}x^{p-1}f^p(x)\times 4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]dx\\
&=4\arctan^2\left[\sqrt{\frac{\gamma}{4-\gamma}}\right]\int_0^{+\infty}x^{p-1}f^p(x)dx.
\end{aligned}\tag{17}$$

For the term M, we proceed in a similar manner, but with the change of variables $v=x/y$ with respect to x, as follows:

$$\begin{aligned}
M &= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{x} \log \left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2} \right] dx \right\} dy \\
&= \int_0^{+\infty} y^{q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{y}{x} \log \left[\frac{(x/y+1)^2}{(x/y)^2+(2-\gamma)x/y+1} \right] \frac{1}{y} dx \right\} dy \\
&= \int_0^{+\infty} y^{q-1} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{v} \log \left[\frac{(v+1)^2}{v^2+(2-\gamma)v+1} \right] dv \right\} dy \\
&= \int_0^{+\infty} y^{q-1} g^q(y) \times 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}} \right] dy \\
&= 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}} \right] \int_0^{+\infty} y^{q-1} g^q(y) dy. \qquad (18)
\end{aligned}$$

Combining Equations (16), (17) and (18), and using the identity $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \log \left[\frac{(x+y)^2}{x^2+(2-\gamma)xy+y^2} \right] f(x) g(y) dx dy \\
&\leq \left\{ 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}} \right] \int_0^{+\infty} x^{p-1} f^p(x) dx \right\}^{1/p} \times \\
&\left\{ 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}} \right] \int_0^{+\infty} y^{q-1} g^q(y) dy \right\}^{1/q} \\
&= 4 \arctan^2 \left[\sqrt{\frac{\gamma}{4-\gamma}} \right] \left[\int_0^{+\infty} x^{p-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q-1} g^q(y) dy \right]^{1/q}.
\end{aligned}$$

This concludes the proof of Proposition 3.10. □

As a special example, if we take $\gamma = 1$, using the identity $\arctan[1/\sqrt{3}] = \pi/6$, then we get

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \log \left[\frac{(x+y)^2}{x^2+xy+y^2} \right] f(x) g(y) dx dy \\
&\leq \frac{\pi^2}{9} \left[\int_0^{+\infty} x^{p-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q-1} g^q(y) dy \right]^{1/q},
\end{aligned}$$

where $\pi^2/9 \approx 1.096622$.

This is a novel logarithmic HardyâHilbert integral inequality. Once again, we emphasize the elegant simplicity and originality of this inequality, which includes a constant factor dependent on π.

4 Conclusion

This article introduced new integral formulas, including some of the logarithmic type. Notably, these formulas are not listed in the reference book [1]. They are characterized by their simplicity and tractability.

Based on this, we have developed new weighted Hölder-type and Hardy-Hilbert-type integral inequalities that differ significantly from existing results in terms of both form and structure. In particular, they offer greater flexibility by including an adjustable parameter. With this new material, we complete the collection of the logarithmic Hardy-Hilbert-type integral inequalities established in [2, 10–12, 14].

There are several possible directions for future research. An obvious extension would be to explore multidimensional analogues of these inequalities. Another approach would be to apply the new inequalities to problems in areas such as harmonic analysis, information theory, and mathematical physics. It would also be interesting to study optimality conditions and sharpen the involved constants. Finally, integrating these results into the framework of functional spaces, such as Orlicz or Lorentz spaces, could lead to a deeper understanding and wider applications.

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