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Interpolative Berinde Weak Mapping Theorem on Partial Metric Spaces

Clement Boateng Ampadu

31 Carrolton Road, Boston, MA 02132-6303, USA

e-mail: profampadu@gmail.com

Abstract

In this paper we introduce the notion of an interpolative Berinde weak operator in partial metric spaces. Additionally, we give an existence theorem for such operators in partial metric spaces. Finally, in support of the existence theorem, we provide an example.

1 Introduction and Preliminaries

Theorem 1.1. [1] *Let (X,d) be a metric space. Suppose $T : X \mapsto X$ satisfies*

$$d(Tx,Ty) \leq kd(x,y)$$

for all $x,y \in X$ and $k \in [0,1)$. Then T has a unique fixed point in X.

Theorem 1.2. [2] *Let (X,d) be a metric space. Suppose $T : X \mapsto X$ satisfies*

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)]$$

for all $x,y \in X$ and $k \in [0, \frac{1}{2})$. Then T has a unique fixed point in X.

Theorem 1.3. [3] *Let (X,d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Kannan type contraction, that is, there are constants $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda d(x,Tx)^{\alpha} d(y,Ty)^{1-\alpha}$$

for all $x,y \in X \backslash Fix(T)$. Then T has a unique fixed point in X.

Theorem 1.4. ([4]- [8]) *Let (X,d) be a metric space. Suppose $T : X \mapsto X$ is a Reich-Rus-Ciric contraction, that is, there exists $\lambda \in [0, \frac{1}{3})$ such that*

$$d(Tx,Ty) \leq \lambda[d(x,y) + d(x,Tx) + d(y,Ty)]$$

for all $x,y \in X$. Then T has a unique fixed point in X.

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From now on we review some basics in partial metric spaces.

Definition 1.5. [9] *Let X be a nonempty set. A function $p : X \times X \mapsto [0, \infty)$ is said to be a partial metric if the following conditions hold for each $x, y, z \in X$*

(a) $x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$;

(b) $p(x, x) \leq p(x, y)$;

(c) $p(x, y) = p(y, x)$;

(d) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, (X, p) is said to be a partial metric space

Example 1.6. [10] *Let (X, p) be a partial metric space. The function $\rho_p : X \times X \mapsto [0, \infty)$ defined as*

$$\rho_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a standard metric on X.

Definition 1.7. ([9], [11]- [20]) *Let (X, p) be a partial metric space. We say that*

(a) *A sequence $\{x_n\}$ converges to a limit x, if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.*

(b) *A sequence $\{x_n\}$ is fundamental or Cauchy if $\lim_{n,m\to\infty} p(x_m, x_n)$ exists and is finite.*

(c) *A partial metric space (X, p) is complete if each fundamental sequence $\{x_n\}$ converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

(d) *A mapping $F : X \mapsto X$ is continuous at a point $x_0 \in X$, if for each $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0; \delta)) \subseteq B_p(F(x_0), \epsilon)$.*

Lemma 1.8. [9] *Let p be a partial metric on a nonempty set X, and ρ_p be the corresponding standard metric on the same set X.*

(a) *A sequence $\{x_n\}$ is fundamental in (X, p) if and only if it is a fundamental sequence in (X, ρ_p).*

(b) *A partial metric space (X, p) is complete if and only if the corresponding standard metric space (X, ρ_p) is complete. Moreover,*

$$\lim_{n\to\infty} \rho_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

(c) *If $x_n \to y$ as $n \to \infty$ in a partial metric space (X,p) with $p(y,y) = 0$, then we have*

$$\lim_{n\to\infty} p(x_n, z) = p(y, z) \text{ for every } z \in X.$$

Definition 1.9. [21] *Let (X,d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx,Ty) \le \lambda d(x,y)^{\alpha} d(x,Tx)^{1-\alpha}$$

where $\lambda \in [0,1)$ and $\alpha \in (0,1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows

Definition 1.10. [21] *Let (X,d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx,Ty) \le \lambda d(x,y)^{\frac{1}{2}} d(x,Tx)^{\frac{1}{2}}$$

where $\lambda \in [0,1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Theorem 1.11. [21] *Let (X,d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator. If (X,d) is complete, then the fixed point of T exists.*

2 Main Result

Definition 2.1. *Let (X,p) be a partial metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$p(Tx,Ty) \le \lambda p(x,y)^{\alpha} p(x,Tx)^{1-\alpha}$$

where $\lambda \in [0,1)$ and $\alpha \in (0,1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Theorem 2.2. *Let (X,p) be a partial metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator, then T has a fixed point in X.*

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_n = T^n(x_0)$ for each positive integer n. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T, and the proof is finished. From now on we assume that $x_n \ne x_{n+1}$ for each $n \ge 0$. From Definition 2.1, observe we have

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Tx_n, Tx_{n-1}) \\ &\le \lambda p(x_n, x_{n-1})^{\alpha} p(x_n, Tx_n)^{1-\alpha} \\ &= \lambda p(x_n, x_{n-1})^{\alpha} p(x_n, x_{n+1})^{1-\alpha}. \end{aligned}$$

From the above inequality, we deduce that

$$p(x_{n+1}, x_n)^\alpha \leq \lambda p(x_n, x_{n-1})^\alpha.$$

Thus, it follows that $\{p(x_{n-1}, x_n)\}$ is a non-increasing sequence with non-negative terms. Thus, there is a non-negative constant l such that $\lim_{n\to\infty} p(x_{n-1}, x_n) = l$. We claim that $l = 0$. Since $p(x_{n+1}, x_n)^\alpha \leq \lambda p(x_n, x_{n-1})^\alpha$, we dedeuce that

$$p(x_n, x_{n+1}) \leq \lambda p(x_{n-1}, x_n) \leq \lambda^n p(x_0, x_1).$$

Since $\lambda < 1$, if we take limits as $n \to \infty$ in the above inequality, we deduce that $l = 0$ Now we show that $\{x_n\}$ is a fundamental (Cauchy) sequence. Since $p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1)$, then using the triangular inequality we deduce the following

$$\begin{aligned} p(x_n, x_{n+r}) &\leq p(x_n, x_{n+1}) + \cdots + p(x_{n+r-1}, x_{n+r}) \\ &\leq \lambda^n p(x_0, x_1) + \cdots + \lambda^{n+r-1} p(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} p(x_0, x_1). \end{aligned}$$

If we take limits in the above inequality as $n \to \infty$, we conclude that $\{x_n\}$ is a fundamental sequence in (X, p). By Lemma 1.8, $\{x_n\}$ is also Cauchy in (X, ρ_p). Since (X, p) is complete, (X, ρ_p) is also complete. Hence there is $x \in X$ such that

$$p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0$$

which implies that

$$\lim_{n\to\infty} \rho_p(x, x_n) = 0.$$

Now we show that x is a fixed point of T. For this, assume that $x \neq Tx$, so $p(x, Tx) > 0$. Since $x_n \neq Tx_n$ for each $n \geq 0$, from Definition 2.1, we deduce the following

$$\begin{aligned} p(x_{n+1}, Tx) &= p(Tx_n, Tx) \\ &\leq \lambda p(x_n, x)^\alpha p(x_n, Tx_n)^{1-\alpha} \\ &= \lambda p(x_n, x)^\alpha p(x_n, x_{n+1})^{1-\alpha}. \end{aligned}$$

Now taking limits in the above inequality as $n \to \infty$, we deduce that $p(x, Tx) = 0$, so $x = Tx$, which is a contradiction.Thus, $x = Tx$, and the proof is finished. □

Example 2.3. *Let $X = \{1, 3, 4, 7\}$ be a set endowed with the classical partial metric $p(x, y) = \max\{x, y\}$, that is,*
$p(1,1) = 1,\ p(1,3) = 3,\ p(1,4) = 4,\ p(1,7) = 7,\ p(3,1) = 3,\ p(3,3) = 3,\ p(3,4) = 4,\ p(3,7) = 7,$
$p(4,1) = 4,\ p(4,3) = 4,\ p(4,4) = 4,\ p(4,7) = 7,\ p(7,1) = 7,\ p(7,3) = 7,\ p(7,4) = 7,\ p(7,7) = 7.$

We define a self mapping T on X by $T(1) = 1$, $T(3) = 3$, $T(4) = 1$, $T(7) = 3$.

Let $\alpha = \frac{1}{2}$ and $\lambda = \frac{7}{10}$. Let $x, y \in X \backslash Fix(T)$, then $(x, y) \in \{(4,7),(7,4),(4,4),(7,7)\}$. Without loss of generality, we have

Case 1: x=y=4

$$1 = p(1,1) = p(T4, T4) < \frac{28}{10} = \frac{7}{10} p(4,4)^{\frac{1}{2}} p(4, T4)^{\frac{1}{2}}.$$

Case 2: x=y=7

$$3 = p(3,3) = p(T7, T7) < \frac{49}{10} = \frac{7}{10} p(7,7)^{\frac{1}{2}} p(7, T7)^{\frac{1}{2}}.$$

Case 3: x=4 and y=7

$$3 = p(1,3) = p(T4, T7) < \frac{7}{10} p(4,7)^{\frac{1}{2}} p(4, T4)^{\frac{1}{2}} \approx 3.704.$$

Thus the self-mapping T is an interpolative Berinde weak contraction, and 1, 3 *are the desired fixed points.*

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