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Gegenbauer Polynomials for a Subfamily of Bi-univalent Functions

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Abstract

We study a subfamily of bi-univalent and regular functions in the open unit disk subordinate to Gegenbauer polynomials. For functions in the defined subfamily, we derive initial coefficients bounds. Additionally, the Fekete-Szegö problem is handled for the elements of the defined subfamily. We also discuss relevant connections to previous findings and several fresh outcomes are shown to follow.

1 Preliminaries

A productive area of mathematics within complex analysis is Geometric Function Theory (GFT). In recent years, this sub-branch has succeeded in drawing researchers' attention. let $\mathfrak{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, where $\mathbb{C}$

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is the complex plane, be the open unit disc. The class of regular functions ϕ in $\mathfrak{U}$ is identified by $\mathcal{A}$ and elements of $\mathcal{A}$ are of the form

$$\phi(\varsigma) = \varsigma + d_2\varsigma^2 + d_3\varsigma^3 + \cdots = \varsigma + \sum_{j=2}^{\infty} d_j\varsigma^j, \quad \varsigma \in \mathfrak{U}, \tag{1.1}$$

and let $\mathcal{S} = \{\phi \in \mathcal{A} : \phi$ is univalent in $\mathfrak{U}\}$. In [2], Biebereach conjectured that $|d_j| \leq j, j \geq 2$ for every function $\phi \in \mathcal{S}$. Numerous new subclasses of $\mathcal{S}$ were defined to settle the Biebereach conjecture and a number of results were established. Reserchers worked on this conjecture's proof for many years and finally, Luis De Branges solved this conjecture for every $j \geq 2$ in [7]. Another problem in GFT is Fekete-Szegö functional $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for every function $\phi \in \mathcal{S}$ [14]. Numerous papers on the aforementioned problem for functions that belong to subclasses of $\mathcal{S}$ have been published by well-known researchers. One of the remarkable subclass of $\mathcal{S}$ is bi-univalent function class σ. In his work [17], Levin introduced the idea of σ of bi-univalent functions. Let ϕ represents these analytic functions, where ϕ and $\phi^{-1} = \psi$ are both univalent in $\mathfrak{U}$. The renowned Koebe theorem (see [9]) states that, each function $\phi \in \mathcal{S}$ of the form (1.1) has an inverse given by

$$\phi^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots = \psi(w) \tag{1.2}$$

satisfying $\varsigma = \psi(\phi(\varsigma))$ and $w = \phi(\psi(w))$, $|w| < r_0(\phi)$, $1/4 \leq r_0(\phi)$, $\varsigma, w \in \mathfrak{U}$. The class σ is not an empty set since the functions $\frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $-\log(1-\varsigma)$ and $\frac{\varsigma}{1-\varsigma}$ are functions in the σ family. However, $\varsigma - \frac{\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$, and the Koebe function $\frac{\varsigma}{(1-\varsigma)^2}$ are not elements of σ, even though they are in $\mathcal{S}$. For a succinct examination and to learn about some of the traits of the σ family, see [3, 4, 18, 33]. The article by Srivastava and his co-authorss [23] triggered the recent surge in research on the bi-univalent function family. Since this article brought the topic back to life, many researchers have investigated a number of intriguing special families of σ; see [5, 6, 8, 10, 11, 34] and the citation given in these papers.

In many fields, including number theory, numerical analysis, combinatorics, computer science, physics, and engineering, special polynomials like Faber, Lucas, Chebyshev, Horadam, Bernoulli, Lucas-Lehmer, Pell-Lucas, Fibonacci, and their generalizations are crucial. Researchers have recently focused attention on a specific type of polynomials called Gegenbauer polynomials (GP).

Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{N} = \{1, 2, 3, \cdots\}$. Recently, Kiepiela et al. [16] have examined the GP $C_j^{\alpha}(x)$. It is also known as ultraspherical polynomials. They can be defined on $[-1, 1]$ by the recurrence relation

$$C_j^{\alpha}(x) = \frac{2x(j+\alpha-1)C_{j-1}^{\alpha}(x) + (j+2\alpha-1)C_{j-2}^{\alpha}(x)}{j}, \quad j \in \mathbb{N}\backslash\{1\}, \tag{1.3}$$

with

$$C_0^{\alpha}(x) = 1 \text{ and } C_1^{\alpha}(x) = 2\alpha x. \tag{1.4}$$

It is evident from (1.3) and (1.4) that

$$C_2^{\alpha}(x) = 2\alpha(1+\alpha)x^2 - \alpha. \tag{1.5}$$

For $\alpha \in \mathbb{R}\backslash\{0\}$, a generating function of the sequence $C_j^{\alpha}(x)$, $j \in \mathbb{N}$, is defined by (see [1])

$$\mathfrak{H}_{\alpha}(x,\varsigma) = \sum_{j=0}^{\infty} C_j^{\alpha}(x)\varsigma^j = \frac{1}{(1-2x\varsigma+\varsigma)^{\alpha}}. \tag{1.6}$$

$C_j^{\frac{1}{2}}(x)$: the Legendre polynomials and $C_j^{1}(x)$: the second kind Chebyshev polynomials, are the two cases of $C_j^{\alpha}(x)$ (see [1]).

The focus in the last two decades was on functions that belong to a specific σ subfamily and are subordinate to known number sequences or special polynomials. Several researchers have found coefficient estmates and Fekete-Szegö functional $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for elements of σ subclasses that are subordinate to number sequences or special polynomials (Refer to [12,13,15,19,20,22,24,25,27,29,31,32,36]). Researchers have recently focused attention on interesting findings about coefficient estimates and the Fekete-Szegö functional for elements of particular subfamilies of σ related to GP [26,30,35].

We introduce a subfamily of σ that is subordinate to a GP: $\mathfrak{T}_{\sigma}(\varrho,\delta,\varkappa)$, which is motivated by the Fekete-Szegö functional on particular subfamilies of σ and the previously mentioned patterns in coefficient-related problems.

For $\mathfrak{a}_1$, $\mathfrak{a}_2 \in \mathcal{A}$ analytic in $\mathfrak{U}$, $\mathfrak{a}_1$ is subordinate to $\mathfrak{a}_2$, if there is a Schwarz function $\theta(\varsigma)$ that is analytic in $\mathfrak{U}$ with $\theta(0) = 0$ and $|\theta(\varsigma)| < 1$, such that $\mathfrak{a}_1(\varsigma) = \mathfrak{a}_2(\theta(\varsigma))$, $\varsigma \in \mathfrak{U}$ [9]. This subordination is symbalized as $\mathfrak{a}_1 \prec \mathfrak{a}_2$ or $\mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma)$ $(\varsigma \in \mathfrak{U})$. In case, if $\mathfrak{a}_2 \in \mathcal{S}$, then

$$\mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma) \;\Leftrightarrow\; \mathfrak{a}_1(0) = \mathfrak{a}_2(0) \quad \text{and} \quad \mathfrak{a}_1(\mathfrak{U}) \subset \mathfrak{a}_2(\mathfrak{U}).$$

Definition 1.1. Let $\varrho \geq 0, 0 < \delta \leq 1$, $\frac{1}{2} < x \leq 1$, and $\alpha \in \mathbb{R}\backslash\{0\}$. If $\phi \in \sigma$ satisfies

$$\frac{1}{2}\left(\left(\frac{(\varsigma\phi'(\varsigma)+\varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)}\right) + \left(\frac{(\varsigma\phi'(\varsigma)+\varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)}\right)^{\frac{1}{\delta}}\right) \prec \mathfrak{H}_{\alpha}(x,\varsigma), \tag{1.7}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w)+\varrho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\delta}}\right) \prec \mathfrak{H}_{\alpha}(x,w), \tag{1.8}$$

then we say that $\phi \in \mathfrak{T}_{\sigma}(\varrho,\delta,x)$, where $\mathfrak{H}_{\alpha}(x,\varsigma)$ is as given by (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\varsigma, w \in \mathfrak{U}$.

In Section 2, we find estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$, for functions in the classes $\mathfrak{T}_{\sigma}(\varrho,\delta,x)$. In Section 3, a number of new findings are presented as a result, and we also talk about pertinent connections to earlier findings

2 Principal Findings

For any function $\phi \in \mathfrak{T}_\sigma(\varrho, \delta, x)$, we determine the coefficient-related estimates.

Theorem 2.1. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{T}_\sigma(\varrho, \delta, x)$, ($\varrho \geq 0, 0 < \delta \leq 1, \frac{1}{2} < x \leq 1$, and $\alpha \in \mathbb{R}\backslash\{0\}$), then*

$$|d_2| \leq \frac{2\delta|\alpha|x\sqrt{2x}}{\sqrt{|(\delta+1)^2(1+2\varrho)^2(1-2x^2) + (2\delta(\delta+1)(5\varrho+1) - (\delta^2+4\delta-1)(2\varrho+1)^2)2\alpha x^2|}}, \tag{2.1}$$

$$|d_3| \leq \left(\frac{2\delta\alpha x}{(\delta+1)(1+2\varrho)}\right)^2 + \frac{2\delta|\alpha|x}{3(\delta+1)(1+3\varrho)}, \tag{2.2}$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{2\delta|\alpha|x}{3(\delta+1)(1+3\varrho)} & ; |1-\xi| \leq \daleth \\ \frac{8\delta^2|\alpha|^2x^3\,|1-\xi|}{|(\delta+1)^2(1+2\varrho)^2(1-2x^2)+(2\delta(\delta+1)(5\varrho+1)-(\delta^2+4\delta-1)(2\varrho+1)^2)2\alpha x^2|} & ; |1-\xi| \geq \daleth, \end{cases} \tag{2.3}$$

where

$$\daleth = \left|\frac{(\delta+1)^2(1+2\varrho)^2(1-2x^2) + (2\delta(\delta+1)(5\varrho+1) - (\delta^2+4\delta-1)(2\varrho+1)^2)2\alpha x^2}{12\alpha\delta(\delta+1)(1+3\varrho)x^2}\right|. \tag{2.4}$$

Proof. Let $\phi \in \mathfrak{T}_\sigma(\varrho, \delta, x)$. Then, for two holomorphic functions $\mathfrak{M}$ and $\mathfrak{N}$ with $\mathfrak{M}(0) = 0 = \mathfrak{N}(0)$, $|\mathfrak{M}(\varsigma)| < 1$, and $|\mathfrak{N}(w)| < 1$, $\varsigma, w \in \mathfrak{U}$ and on account of Definition 1.1 we can write

$$\frac{1}{2}\left(\left(\frac{(\varsigma\phi'(\varsigma) + \varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)}\right) + \left(\frac{(\varsigma\phi'(\varsigma) + \varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)}\right)^{\frac{1}{\delta}}\right) = \mathfrak{H}_\alpha(x, \mathfrak{M}(\varsigma)), \tag{2.5}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w) + \varrho w^2\psi''(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w) + \varrho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\delta}}\right) = \mathfrak{H}_\alpha(x, \mathfrak{N}(w)), \tag{2.6}$$

A few basic mathematical methods allow us to write equations (2.5) and (2.6) as

$$\frac{1}{2}\left(\left(\frac{(\varsigma\phi'(\varsigma) + \varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)}\right) + \left(\frac{(\varsigma\phi'(\varsigma) + \varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)}\right)^{\frac{1}{\delta}}\right) = 1 + \left(\frac{\delta+1}{\delta}\right)(1+2\varrho)d_2\varsigma$$
$$+ \left(\left(\frac{\delta+1}{\delta}\right)(3(1+3\varrho)d_3 - 2(1+2\varrho)d_2^2) + \left(\frac{1-\delta}{\delta^2}\right)(1+2\varrho)^2 d_2^2\right)\varsigma^2 + \cdots \tag{2.7}$$

$$\mathfrak{H}_\alpha(x, \mathfrak{M}(\varsigma)) = 1 + C_1^\alpha(x)\mathfrak{m}_1\varsigma + \left[C_1^\alpha(x)\mathfrak{m}_2 + C_2^\alpha(x)\mathfrak{m}_1^2\right]\varsigma^2 + ..., \tag{2.8}$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w) + \varrho w^2\psi''(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w) + \varrho w^2\psi''(w))'}{\psi'(w)}\right)^{\frac{1}{\delta}}\right) = 1 - \left(\frac{\delta+1}{\delta}\right)(1+2\varrho)d_2 w$$

$$+\left(\left(\frac{\delta+1}{\delta}\right)(2(2+7\varrho)d_2^2-3(1+3\varrho)d_3)+\left(\frac{1-\delta}{\delta^2}\right)(1+2\varrho)^2d_2^2\right)w^2+\cdots \tag{2.9}$$

$$\mathfrak{H}_\alpha(x,\mathfrak{N}(w))=1+C_1^\alpha(x)\mathfrak{n}_1w+\left[C_1^\alpha(x)\mathfrak{n}_2+C_2^\alpha(x)\mathfrak{n}_1^2\right]w^2+\ldots \tag{2.10}$$

It is known that if $|\mathfrak{M}(\varsigma)|=|\mathfrak{m}_1\varsigma+\mathfrak{m}_2\varsigma^2+\cdots|<1$, $\varsigma\in\mathfrak{U}$ and $|\mathfrak{N}(\varsigma)|=|\mathfrak{n}_1w+\mathfrak{n}_2w^2+\cdots|<1$, $w\in\mathfrak{U}$, then

$$|\mathfrak{m}_i|\leq 1,\ \text{and}\ |\mathfrak{n}_i|\leq 1,(i\in\mathbb{N}). \tag{2.11}$$

By comparing the terms of the same degree in (2.7) and (2.9), we arrive at the following conclusions due to (2.5).

$$\left(\frac{\delta+1}{\delta}\right)(1+2\varrho)d_2=C_1^\alpha(x)\mathfrak{m}_1, \tag{2.12}$$

$$\left(\frac{\delta+1}{\delta}\right)(3(1+3\varrho)d_3-2(1+2\varrho)d_2^2)+\left(\frac{1-\delta}{\delta^2}\right)(1+2\varrho)^2d_2^2=C_1^\alpha(x)\mathfrak{m}_2+C_2^\alpha(x)\mathfrak{m}_1^2. \tag{2.13}$$

Likewise, because of equality (2.6), we compare terms of the same degree in (2.8) and (2.10) to arrive at our conclusion.

$$-\left(\frac{\delta+1}{\delta}\right)(1+2\varrho)d_2=C_1^\alpha(x)\mathfrak{n}_1, \tag{2.14}$$

and

$$\left(\frac{\delta+1}{\delta}\right)(2(2+7\varrho)d_2^2-3(1+3\varrho)d_3)+\left(\frac{1-\delta}{\delta^2}\right)(1+2\varrho)^2d_2^2=C_1^\alpha(x)\mathfrak{n}_2+C_2^\alpha(x)\mathfrak{n}_1^2. \tag{2.15}$$

From (2.12) and (2.14), we get

$$\mathfrak{m}_1=-\mathfrak{n}_1, \tag{2.16}$$

and

$$2\left(\frac{\delta+1}{\delta}\right)^2(1+2\varrho)^2d_2^2=(\mathfrak{m}_1^2+\mathfrak{n}_1^2)(C_1^\alpha(x))^2. \tag{2.17}$$

Addition of (2.13) and (2.15) yield

$$\left(\left(\frac{1+\delta}{\delta}\right)(1+5\varrho)+\left(\frac{1-\delta}{\delta^2}\right)(1+2\varrho)^2\right)2d_2^2=C_1^\alpha(x)(\mathfrak{m}_2+\mathfrak{n}_2)+C_2^\alpha(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.18}$$

Replacing $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.17) in (2.18) we get

$$d_2^2=\frac{\delta^2(C_1^\alpha(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2(\delta(\delta+1)(1+5\varrho)+(1-\delta)(1+2\varrho)^2)(C_1^\alpha(x))^2-(\delta+1)^2(1+2\varrho)^2C_2^\alpha(x)}. \tag{2.19}$$

Utilizing (1.4) and (1.5) for $C_1^\alpha(x)$ and $C_2^\alpha(x)$, respectively and applying (2.11) to $\mathfrak{m}_2,\mathfrak{n}_2$ produces (2.1).

From (2.13) we subtract (2.15) to get the bound on $|d_3|$:

$$d_3=d_2^2+\frac{\delta C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{6(\delta+1)(1+3\varrho)}. \tag{2.20}$$

If we replace d_2^2 from (2.17) in (2.20) we get

$$d_3 = \frac{\delta^2 (C_1^\alpha(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_2^2)}{2(\delta+1)^2(1+2\varrho)^2} + \frac{\delta C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{6(\delta+1)(1+3\varrho)}. \tag{2.21}$$

We deduce (2.2) from (2.21), utilizing (1.4) and (1.5) for $C_1^\alpha(x)$ and $C_2^\alpha(x)$, respectively and applying (2.11) to $\mathfrak{m}_2, \mathfrak{n}_2$. Finally, we compute the bound on $|d_3 - \xi d_2^2|$ using the values of d_2^2 and d_3 from (2.19) and (2.20), respectively. Consequently, we have

$$d_3 - \xi d_2^2 = \frac{|C_1^\alpha(x)|}{2}\left|\left(\frac{\delta}{3(\delta+1)(1+3\varrho)} + \mathcal{V}(\xi,x)\right)\mathfrak{m}_2 - \left(\frac{\delta}{3(\delta+1)(1+3\varrho)} - \mathcal{V}(\xi,x)\right)\mathfrak{n}_2\right|,$$

where

$$\mathcal{V}(\xi,x) = \frac{\delta^2(C_1^\alpha(x))^2(1-\xi)}{(\delta(\delta+1)(1+5\varrho)+(1-\delta)(1+2\varrho)^2)(C_1^\alpha(x))^2 - (\delta+1)^2(1+2\varrho)^2 C_2^\alpha(x)}.$$

Clearly

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{\delta|C_1^\alpha(x)|}{3(\delta+1)(1+3\varrho)} & ; \ |\mathcal{V}(\xi,x)| \leq \frac{\delta}{3(\delta+1)(1+3\varrho)} \\ |C_1^\alpha(x)||\mathcal{V}(\xi,x)| & ; \ |\mathcal{V}(\xi,x)| \geq \frac{\delta}{3(\delta+1)(1+3\varrho)}. \end{cases} \tag{2.22}$$

We derive (2.3) from (2.22), where $\daleth$ is the same as in (2.4). □

The following is obtained by applying $\xi = 1$ in Theorem 2.1:

Corollary 2.1. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{T}_\sigma(\varrho,\delta,x)$, ($\varrho \geq 0, 0 < \delta \leq 1, \frac{1}{2} < x \leq 1$, and $\alpha \in \mathbb{R}\backslash\{0\}$), then $|d_3 - d_2^2| \leq \frac{2\delta|\alpha|x}{3(\delta+1)(1+3\varrho)}$.*

3 Specific Instances

Here are some specific instances of our theorem proved in Section 1.

Example 3.1. Letting $\varrho = 0$ in the class $\mathfrak{T}_\sigma(\varrho,\delta,x)$, we get a subclass $\mathfrak{C}_\sigma(\delta,x) \equiv \mathfrak{T}_\sigma(0,\delta,x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2}\left(\left(\frac{(\varsigma\phi'(\varsigma))'}{\phi'(\varsigma)}\right) + \left(\frac{(\varsigma\phi'(\varsigma))'}{\phi'(\varsigma)}\right)^{\frac{1}{\delta}}\right) \prec \mathfrak{H}_\alpha(x,\varsigma),$$

and

$$\frac{1}{2}\left(\left(\frac{(w\psi'(w))'}{\psi'(w)}\right) + \left(\frac{(w\psi'(w))'}{\psi'(w)}\right)^{\frac{1}{\delta}}\right) \prec \mathfrak{H}_\alpha(x,w),$$

where $0 < \delta \leq 1, \frac{1}{2} < x \leq 1$, $\alpha \in \mathbb{R}\backslash\{0\}$, $\mathfrak{H}_\alpha(x,\varsigma)$ is as given by (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\varsigma, w \in \mathfrak{U}$. .

Corollary 3.1. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{C}_\sigma(\delta, x)$, then*

$$|d_2| \leq 2\delta|\alpha|x\sqrt{\frac{2x}{|(\delta+1)^2(1-2x^2)+(\delta-1)^2 2\alpha x^2|}}, \quad |d_3| \leq \left(\frac{2\delta\alpha x}{\delta+1}\right)^2 + \frac{2\delta|\alpha|x}{3(\delta+1)},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{2\delta|\alpha|x}{3(\delta+1)} & ; |1-\xi| \leq \left|\frac{(\delta+1)^2(1-2x^2)+(\delta-1)^2 2\alpha x^2}{12\alpha\delta(\delta+1)x^2}\right| \\ \frac{8\delta^2|\alpha|^2x^3\,|1-\xi|}{|(\delta+1)^2(1-2x^2)+(\delta-1)^2 2\alpha x^2|} & ; |1-\xi| \geq \left|\frac{(\delta+1)^2(1-2x^2)+(\delta-1)^2 2\alpha x^2}{12\alpha\delta(\delta+1)x^2}\right|. \end{cases}$$

The following inequality is obtained by allowing $\xi = 1$ in Corollary 3.1.

Corollary 3.2. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{C}_\sigma(\delta, x)$, then $|d_3 - d_2^2| \leq \frac{2\delta|\alpha|x}{3(\delta+1)}$.*

Example 3.2. Letting $\delta = 1$ in the class $\mathfrak{T}_\sigma(\varrho, \delta, x)$, we get a subclass $\mathfrak{F}_\sigma(\varrho, x) \equiv \mathfrak{T}_\sigma(\varrho, 1, x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{(\varsigma\phi'(\varsigma) + \varrho\varsigma^2\phi''(\varsigma))'}{\phi'(\varsigma)} \prec \mathfrak{H}_\alpha(x, \varsigma),$$

and

$$\frac{(w\psi'(w) + \varrho w^2\psi''(w))'}{\psi'(w)} \prec \mathfrak{H}_\alpha(x, w),$$

where $\varrho \geq 0, \frac{1}{2} < x \leq 1$, $\alpha \in \mathbb{R}\backslash\{0\}$, $\mathfrak{H}_\alpha(x, \varsigma)$ is as given by (1.3), $\psi(w) = \phi^{-1}(w)$ is as in (1.2), and $\varsigma, w \in \mathfrak{U}$.

Corollary 3.3. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{F}_\sigma(\varrho, x)$, then*

$$|d_2| \leq |\alpha|x\sqrt{\frac{2x}{|(2\varrho+1)^2(1-2x^2)+\varrho(1-4\varrho)2\alpha x^2|}}, \quad |d_3| \leq \frac{\alpha^2x^2}{(1+2\varrho)^2} + \frac{|\alpha|x}{3(1+3\varrho)},$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\alpha|x}{3(1+3\varrho)} & ; |1-\xi| \leq \left|\frac{(2\varrho+1)^2(1-2x^2)+\varrho(1-4\varrho)2\alpha x^2}{24\alpha(1+3\varrho)x^2}\right| \\ \frac{2|\alpha|^2x^3\,|1-\xi|}{|(2\varrho+1)^2(1-2x^2)+\varrho(1-4\varrho)2\alpha x^2|} & ; |1-\xi| \geq \left|\frac{(2\varrho+1)^2(1-2x^2)+\varrho(1-4\varrho)2\alpha x^2}{24\alpha(1+3\varrho)x^2}\right|. \end{cases}$$

Remark 3.1. If $\varrho = 0$ in Corollary 3.3, we obtain Theorem 3.1 and Theorem 5.1 of [1], correcting the bound of $|d_2|$.

Using $\xi = 1$ in Corollary 3.3, we obtain the inequality that follows:

Corollary 3.4. *If a function $\phi \in \sigma$ is a member of the family $\mathfrak{F}_\sigma(\varrho, x)$, then $|d_3 - d_2^2| \leq \frac{|\alpha|x}{3(1+3\varrho)}$.*

Remark 3.2. If $\varrho = 0$ in Corollary 3.4, we obtain Theorem 5.2 of [1].

4 Conclusion

This study contains the upper bounds on $|d_2|$ and $|d_3|$ for functions that belong to the defined σ subclasses associated with BP. Additionally, we have determined the Fekete-Szegö functional $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for functions in these subfamilies. Specialization of the parameters applied to our results, as mentioned in Section 3, produces previously unexplored new results. We conclude our study by pointing out to interested readers that the subclass can be studied for higher order Hankel determinant problems. Interested readers are advised to read these papers [21, 28] and the related references.

Authors' Contributions:

Each author contributed equally to the results derivation and gave their final manuscript approval.

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