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Interpolative Berinde-Meir-Keeler Weak Contraction Mapping Theorem

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Abstract

In this paper, we introduce the notion of an interpolative Berinde-Meir-Keeler weak operator. Additionally, we provide an existence theorem for such operators in metric spaces. An example is given to illustrate the main result.

1 Introduction and Preliminaries

Definition 1.1. *[1] Let (X,d) be a complete metric space. A mapping $T : X \mapsto X$ is said to be a Meir-Keeler contraction on X, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq d(x,y) < \epsilon + \delta \implies d(Tx,Ty) < \epsilon$ for every $x,y \in X$.*

Theorem 1.2. *[1] On a complete metric space (X,d), any Meir-Keeler contraction $T : X \mapsto X$ has a unique fixed point.*

Definition 1.3. *[2] Let (X,d) be a complete metric space. A mapping $T : X \mapsto X$ is said to be an interpolative Kannan type contraction on X, if there exists $k \in [0,1)$ and $\gamma \in (0,1)$, such that, $d(Tx,Ty) \leq kd(x,Tx)^{\gamma} d(y,Ty)^{1-\gamma}$, for every $x,y \in X \setminus Fix(T)$, where $Fix(T) = \{x \in X | Tx = x\}$.*

Theorem 1.4. *[2] On a complete metric space (X,d), any interpolative Kannan contraction $T : X \mapsto X$ has a fixed point.*

Definition 1.5. *[3] Let (X,d) be a complete metric space. A mapping $T : X \mapsto X$ is said to be an interpolative Kannan-Meir-Keeler type contraction on X, if there exists $\gamma \in (0,1)$ such that for every $x,y \in X \setminus Fix(T)$ we have*

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(a) *given* $\epsilon > 0$, *there exists* $\delta > 0$ *so that*

$$\epsilon < d(x,Tx)^{\gamma} d(y,Ty)^{1-\gamma} < \epsilon + \delta \Longrightarrow d(Tx,Ty) \leq \epsilon.$$

(b)

$$d(Tx,Ty) < d(x,Tx)^{\gamma} d(y,Ty)^{1-\gamma}.$$

Theorem 1.6. *[3] On a complete metric space* (X,d), *any interpolative Kannan-Meir-Keeler type contraction* $T : X \mapsto X$ *has a fixed point.*

Definition 1.7. *[4] Let* (X,d) *be a metric space. We say* $T : X \mapsto X$ *is an interpolative Berinde weak operator if it satisfies*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\alpha} d(x,Tx)^{1-\alpha}$$

where $\lambda \in [0,1)$ *and* $\alpha \in (0,1)$, *for all* $x, y \in X$, $x, y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows

Definition 1.8. *[4] Let* (X,d) *be a metric space. We say* $T : X \mapsto X$ *is an interpolative Berinde weak operator if it satisfies*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\frac{1}{2}} d(x,Tx)^{\frac{1}{2}}$$

where $\lambda \in [0,1)$, *for all* $x, y \in X$, $x, y \notin Fix(T)$.

Theorem 1.9. *[4] Let* (X,d) *be a metric space. Suppose* $T : X \mapsto X$ *is an interpolative Berinde weak operator. If* (X,d) *is complete, then the fixed point of* T *exists.*

2 Main Result

Definition 2.1. *Let* (X,d) *be a complete metric space. A mapping* $T : X \mapsto X$ *will be called an interpolative Berinde-Meir-Keeler type contraction on* X, *if there exists* $\gamma \in (0,1)$ *such that for every* $x, y \in X \backslash Fix(T)$ *we have*

(a) *given* $\epsilon > 0$, *there exists* $\delta > 0$ *so that*

$$\epsilon < d(x,y)^{\gamma} d(x,Tx)^{1-\gamma} < \epsilon + \delta \Longrightarrow d(Tx,Ty) \leq \epsilon.$$

(b)

$$d(Tx,Ty) < d(x,y)^{\gamma} d(x,Tx)^{1-\gamma}.$$

Theorem 2.2. *On a complete metric space (X, d), any interpolative Berinde-Meir-Keeler type contraction $T : X \mapsto X$ has a fixed point.*

Proof. Let $x_0 \in X$. Define the sequence $\{x_m\}$ by $x_m = Tx_{m-1} = T^m x_0$. Now observe by (b) of Definition 2.1 we have

$$\begin{aligned} d(x_m, x_{m+1}) &= d(Tx_{m-1}, Tx_m) \\ &< d(x_{m-1}, x_m)^\gamma d(x_{m-1}, Tx_{m-1})^{1-\gamma} \\ &= d(x_{m-1}, x_m). \end{aligned}$$

Therefore the sequence $\{d(x_m, x_{m+1})\}$ is strictly decreasing and since $d(x_m, x_{m+1}) > 0$ for every $m \in \mathbb{N} \cup \{0\}$, it follows that the sequence $\{d(x_m, x_{m+1})\}$ tends to a point $\omega \geq 0$. We claim that $\omega = 0$. Indeed, if we suppose that $\omega > 0$, we can find $N \in \mathbb{N}$ such that

$$\omega < d(x_m, x_{m+1}) < \omega + \delta(\omega)$$

for any $m \geq N$. Then since $\omega < d(x_m, x_{m+1}) < d(x_{m-1}, x_m)^\gamma d(x_{m-1}, x_m)^{1-\gamma}$, keeping in mind (a) of Definition 2,1, it follows that $d(x_m, x_{m+1}) \leq \omega$, for any $m \geq N$. This is a contradiction, and that's why we get $\omega = 0$. Now we show that $\{x_m\}$ is a Cauchy sequence. For this, let $\epsilon > 0$ be fixed and choose $\delta(\epsilon) < \epsilon$. Since $\lim_{m\to\infty} d(x_m, x_{m+1}) = 0$, we can find $l \in \mathbb{N}$ such that $d(x_m, x_{m+1}) < \frac{\epsilon}{2}$ for $m \geq l$, and we claim that

$$d(x_m, x_{m+p}) < \epsilon$$

for any $p \in \mathbb{N}$. Of course the above inequality holds for $p = 1$. Now assume the above inequality holds for some p, we will prove it for $p + 1$. Observe we have

$$\begin{aligned} d(x_m, x_{m+p+1}) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p+1}) \\ &= d(x_m, x_{m+1}) + d(Tx_m, Tx_{m+p}) \\ &< d(x_m, x_{m+1}) + d(x_m, x_{m+p})^\gamma d(x_m, x_{m+1})^{1-\gamma} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore the sequence $\{x_m\}$ is Cauchy, and by the completeness of the space X it follows that there exists $x_* \in X$ such that $\lim_{m\to\infty} x_m = x_*$. Now we will show that $x_* = Tx_*$. Supposing on the contrary that $x_* \neq Tx_*$, observe we have

$$\begin{aligned} 0 &< d(x_*, Tx_*) \\ &\leq d(x_*, x_{m+1}) + d(x_{m+1}, Tx_*) \\ &= d(x_*, x_{m+1}) + d(Tx_m, Tx_*) \\ &< d(x_m, x_*)^\gamma d(x_m, x_{m+1}). \end{aligned}$$

Taking limits in the above inequality as $m \to \infty$, we get $d(x_*, Tx_*) = 0$, that is, x_* is the fixed point of the mapping T. □

Example 2.3. *Let* $X = \mathbb{R}^2$, *and* $\Gamma = \{A, B, C, D\}$, *where* $A = (1, -1)$, $B = (-1, 0)$, $C = (2, -1)$, *and* $D = (2, 0)$. *Let* $d : X \times X \mapsto [0, \infty)$ *be defined as* $d(P, Q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ *for any* $P, Q \in X$, $P = (x_1, x_2)$, $Q = (y_1, y_2)$, *with* $x_1, x_2, y_1, y_2 \in \mathbb{R}$. *Define the mapping* $T : X \mapsto X$ *by*

$$TA = TC = TD = C, TB = D, \text{ and } TP = P \text{ for any } P \in X \backslash \Gamma.$$

We show that T *satisfies the conditions of the above theorem as follows: Let* $\gamma = \frac{1}{2}$. *Now for* $\epsilon < 2^{\frac{1}{4}}$, *with* $\delta = \sqrt{2} - \epsilon$, *we have*

$$\epsilon < 2^{\frac{1}{4}} = \sqrt{d(A, D)d(A, TA)} = \sqrt{d(A, D)d(A, C)} < \sqrt{2} = \epsilon + \delta$$

$\Longrightarrow$

$$d(TA, TD) = d(C, C) = 0 < \epsilon$$

and also

$$0 = d(TA, TD) < \sqrt{d(A, D)d(A, C)} = 2^{\frac{1}{4}}.$$

Now for $\epsilon \geq 2^{\frac{1}{4}}$, *with* $\delta = 2$, *we have*

$$\epsilon < \sqrt{d(A, B)d(A, TA)} = \sqrt{d(A, B)d(A, C)} = \sqrt{5} < \epsilon + \delta$$

$\Longrightarrow$

$$d(TA, TB) = d(C, D) = 1 < \epsilon$$

and also

$$1 = d(TA, TB) < \sqrt{d(A, B)d(A, C)} = \sqrt{5}.$$

Similarly,

$$\epsilon < \sqrt{d(D, B)d(D, TD)} = \sqrt{d(D, B)d(D, C)} = \sqrt{3} < \epsilon + \delta$$

$\Longrightarrow$

$$d(TD, TB) = d(C, D) = 1 < \epsilon$$

and also

$$1 = d(TD, TB) < \sqrt{d(D, B)d(D, C)} = \sqrt{3}.$$

Consequently, by the above theorem, the mapping T *has fixed points, these are* $C = (2, -1)$ *and all* $P \in X \backslash \Gamma$.

References

[1] Meir, A., & Keeler, E. (1969). A theorem on contraction mappings. *Journal of Mathematical Analysis and Applications, 28*, 326-329. https://doi.org/10.1016/0022-247X(69)90031-6

[2] Karapınar, E. (2018). Revisiting the Kannan type contractions via interpolation. *Advances in the Theory of Nonlinear Analysis and Its Applications, 2*(2), 85-87. https://doi.org/10.31197/atnaa.431135

[3] Karapınar, E. (2021). Interpolative Kannan-Meir-Keeler type contraction. *Advances in the Theory of Nonlinear Analysis and Its Applications, 5*(4), 611-614. https://doi.org/10.31197/atnaa.989389

[4] Ampadu, C. B. (2020). Some fixed point theory results for the interpolative Berinde weak operator. *Earthline Journal of Mathematical Sciences, 4*(2), 253-271. https://doi.org/10.34198/ejms.4220.253271