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Strong Differential Subordination Results for Multivalent Analytic Functions Associated with Dziok-Srivastava Operator

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Abstract

In this paper, by making use of the principle of strong subordination, we establish some interesting properties of multivalent analytic functions defined in the open unit disk and closed unit disk of the complex plane associated with Dziok-Srivastava operator.

1. Introduction and Preliminaries

Denote by $\mathcal{H}(U \times \overline{U})$ the family of all analytic functions in $U \times \overline{U}$. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ indicate the open unit disk and the closed unit disk of the complex plane, respectively. For $n \in \mathbb{N} = \{1, 2, ...\}$ and $a \in \mathbb{C}$, let $\mathcal{H}^*[a, n, \zeta] = \{f \in \mathcal{H}(U \times \overline{U}) : f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in U, \zeta \in \overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Also, let $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}) : f(z, \zeta) = z + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in U, \zeta \in \overline{U}\}$, with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_{\zeta}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n+1$.

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A function $f \in \mathcal{H}^*[a, n, \zeta]$ is said to be *starlike* in $U \times \overline{U}$ if

$$\operatorname{Re}\left\{\frac{zf_z'(z, \zeta)}{f(z, \zeta)}\right\} > 0, \quad (z \in U, \zeta \in \overline{U}).$$

Denote the class of all starlike functions in $U \times \overline{U}$ by S_ζ^*.

Similar, $f \in \mathcal{H}^*[a, n, \zeta]$ is said to be *convex* in $U \times \overline{U}$ if

$$\operatorname{Re}\left\{\frac{zf_{z^2}''(z, \zeta)}{f_z'(z, \zeta)} + 1\right\} > 0, \quad (z \in U, \zeta \in \overline{U}).$$

Denote the class of all convex functions in $U \times \overline{U}$ by K_ζ^*.

Definition 1.1 [9]**.** Let $f(z, \zeta)$, $g(z, \zeta)$ be analytic in $U \times \overline{U}$. The function $f(z, \zeta)$ is said to be *strongly subordinate* to $g(z, \zeta)$, written $f(z, \zeta) \prec\prec g(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ such that $f(z, \zeta) = g(w(z), \zeta)$ for all $\zeta \in \overline{U}$.

Remark 1.1 [9]**.**

(1) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0, \zeta) = g(0, \zeta)$ for all $\zeta \in \overline{U}$ and $f(U \times \overline{U}) \subset g(U \times \overline{U})$.

(2) If $f(z, \zeta) = f(z)$ and $g(z, \zeta) = g(z)$, then the strong subordination becomes the usual notion of subordination.

Let $\mathcal{A}_\zeta^*(p)$ denote the subclass of the functions $f(z, \zeta) \in \mathcal{H}(U \times \overline{U})$ of the form:

$$f(z, \zeta) = z^p + \sum_{k=1}^{\infty} a_{p+k}(\zeta) z^{p+k}, \quad p \in \mathbb{N} = \{1, 2, ...\}, z \in U, \zeta \in \overline{U} \tag{1.1}$$

which are analytic and multivalent in $U \times \overline{U}$.

For complex parameters $\alpha_i \in \mathbb{C}, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{0, -1, -2, ...\}$; $1 \le i \le l$, $1 \le j \le m$; $l \le m + 1$; $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}_\zeta^*(p)$. The Dziok-

Srivastava operator $H_p(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m) : \mathcal{A}^*_\zeta(p) \to \mathcal{A}^*_\zeta(p)$ (see for details [3, 6]) is defined by

$$H_p(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m) f(z, \zeta) = z^p + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k k!} a_{p+k}(\zeta) z^{p+k}, \quad (1.2)$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & (k = 0), \\ x(x+1)\cdots(x+k-1) & (k \in \mathbb{N}). \end{cases}$$

In order to make the notation simple, we write $H_p^{l,m}(\alpha_1) = H_p(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)$.

It is easily verified from (1.2) that

$$z(H_p^{l,m}(\alpha_1) f(z, \zeta))'_z = \alpha_1 H_p^{l,m}(\alpha_1 + 1) f(z, \zeta) - (\alpha_1 - p) H_p^{l,m}(\alpha_1) f(z, \zeta). \quad (1.3)$$

We note that special cases of the Dziok-Srivastava operator $H_p^{l,m}(\alpha_1)$ include the Hohlov linear operator [4], the Carlson-Shafer operator [1], the Ruscheweyh derivative operator [11], the Srivastava-Owa fractional operator [10], and many others.

In recent years, many authors obtained various interesting results associated with strong differential subordination and superordination, for example, (see [2, 5, 12, 13, 14, 15]).

We will require the following lemmas in proving our main results:

Lemma 1.1 [8]. *Let* $h(z, \zeta)$ *be a convex function with* $h(0, \zeta) = a$, *for every* $\zeta \in \overline{U}$ *and let* $\gamma \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ *with* $\text{Re}(\gamma) \geq 0$. *If* $p \in \mathcal{H}^*[a, n, \zeta]$ *and*

$$p(z, \zeta) + \frac{1}{\gamma} z p'_z(z, \zeta) \prec\prec h(z, \zeta), \quad (z \in U, \zeta \in \overline{U}), \quad (1.4)$$

then

$$p(z, \zeta) \prec\prec q(z, \zeta) \prec\prec h(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),$$

where $q(z, \zeta) = \dfrac{\gamma}{nz^{\frac{\gamma}{n}}} \displaystyle\int_0^z t^{\frac{\gamma}{n}-1} h(t, \zeta) dt$ *is convex and it is the best dominant of* (1.4).

Lemma 1.2 [7]**.** *Let* $q(z, \zeta)$ *be a convex function in* $U \times \overline{U}$ *for all* $\zeta \in \overline{U}$ *and let* $h(z, \zeta) = q(z, \zeta) + n\delta z q'_z(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$, *where* $\delta > 0$ *and n is a positive integer. If*

$$p(z, \zeta) = q(0, \zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \cdots,$$

is analytic in $U \times \overline{U}$ *and*

$$p(z, \zeta) + \delta z p'_z(z, \zeta) \prec\prec h(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),$$

then

$$p(z, \zeta) \prec\prec q(z, \zeta), \quad (z \in U, \zeta \in \overline{U}),$$

and this result is sharp.

2. Main Results

Theorem 2.1. *Let* $h(z, \zeta)$ *be a convex function such that* $h(0, \zeta) = 1$. *If* $f \in \mathcal{A}^*_\zeta(p)$ *satisfies the strong differential subordination*:

$$\frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))'_z}{p z^{p-1}} \prec\prec h(z, \zeta), \tag{2.1}$$

then

$$\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} \prec\prec q(z, \zeta) \prec\prec h(z, \zeta),$$

where $q(z, \zeta) = \frac{p}{z^p} \int_0^z t^{p-1} h(t, \zeta) dt$ *is convex and it is the best dominant.*

Proof. Suppose that

$$F(z, \zeta) = \frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p}, \quad z \in U, \ \zeta \in \overline{U}. \tag{2.2}$$

Then the function $F(z, \zeta)$ is analytic in $U \times \overline{U}$ and $F(0, \zeta) = 1$.

Simple computations from (2.2), we get

$$F(z, \zeta) + \frac{1}{p} zF_z'(z, \zeta) = \frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))_z'}{pz^{p-1}}. \tag{2.3}$$

Using (2.3), (2.1) becomes

$$F(z, \zeta) + \frac{1}{p} zF_z'(z, \zeta) \prec\prec h(z, \zeta).$$

An application of Lemma 1.1 with $n = 1$, $\gamma = p$ yields

$$\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} \prec\prec q(z, \zeta) = \frac{p}{z^p}\int_0^z t^{p-1} h(t, \zeta) dt \prec\prec h(z, \zeta).$$

By taking $p = 1$ and $h(z, \zeta) = \dfrac{\zeta + (2\lambda - \zeta)z}{1+z}$, $0 \le \lambda < 1$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *If* $f \in \mathcal{A}_\zeta^*(1)$ *satisfies the strong differential subordination*:

$$(H_1^{l,m}(\alpha_1) f(z, \zeta))_z' \prec\prec \frac{\zeta + (2\lambda - \zeta)z}{1+z},$$

then

$$\frac{H_1^{l,m}(\alpha_1) f(z, \zeta)}{z} \prec\prec \frac{1}{z}\int_0^z \frac{\zeta + (2\lambda - \zeta)t}{1+t} dt = 2\lambda - \zeta + \frac{2(\zeta - \lambda)}{z}\ln(1+z).$$

Theorem 2.2. *Let* $h(z, \zeta)$ *be a convex function such that* $h(0, \zeta) = 1$. *If* $0 \le \sigma < p$, $\eta \in \mathbb{C}$ *and* $f \in \mathcal{A}_\zeta^*(p)$ *satisfies the strong differential subordination*:

$$\frac{1-\eta}{1-\sigma}\left(\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} - \sigma\right) + \frac{\eta}{1-\sigma}\left(\frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))_z'}{pz^{p-1}} - \sigma\right) \prec\prec h(z, \zeta), \tag{2.4}$$

then

$$\frac{1}{1-\sigma}\left(\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} - \sigma\right) \prec\prec q(z, \zeta) \prec\prec h(z, \zeta),$$

where $q(z, \zeta) = \frac{p}{\eta} z^{-\frac{p}{\eta}} \int_0^z t^{\frac{p}{\eta}-1} h(t, \zeta) dt$ *is convex and it is the best dominant.*

Proof. Suppose that

$$F(z, \zeta) = \frac{1}{1-\sigma}\left(\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} - \sigma\right), \qquad z \in U, \quad \zeta \in \overline{U}. \tag{2.5}$$

Then the function $F(z, \zeta)$ is analytic in $U \times \overline{U}$ and $F(0, \zeta) = 1.$

Differentiating both sides of (2.5) with respect to z, we have

$$F(z, \zeta) + \frac{\eta}{p} z F_z'(z, \zeta) = \frac{1-\eta}{1-\sigma}\left(\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} - \sigma\right)$$

$$+ \frac{\eta}{1-\sigma}\left(\frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))_z'}{p z^{p-1}} - \sigma\right). \tag{2.6}$$

From (2.4) and (2.6), we get

$$F(z, \zeta) + \frac{\eta}{p} z F_z'(z, \zeta) \prec\prec h(z, \zeta).$$

An application of Lemma 1.1 with $n = 1, \gamma = \frac{p}{\eta}$ yields

$$\frac{1}{1-\sigma}\left(\frac{H_p^{l,m}(\alpha_1) f(z, \zeta)}{z^p} - \sigma\right) \prec\prec q(z, \zeta) = \frac{p}{\eta} z^{-\frac{p}{\eta}} \int_0^z t^{\frac{p}{\eta}-1} h(t, \zeta) dt \prec\prec h(z, \zeta).$$

Theorem 2.3. *Let* $q(z, \zeta)$ *be a convex function such that* $q(0, \zeta) = 1$ *and let h be the function* $h(z, \zeta) = q(z, \zeta) + z q_z'(z, \zeta)$. *If* $f \in \mathcal{A}_\zeta^*(p)$ *satisfies the strong differential subordination*:

$$\left(\frac{z H_p^{l,m}(\alpha_1 + 1) f(z, \zeta)}{H_p^{l,m}(\alpha_1) f(z, \zeta)}\right)_z' \prec\prec h(z, \zeta), \tag{2.7}$$

then

$$\frac{H_p^{l,m}(\alpha_1+1)f(z,\zeta)}{H_p^{l,m}(\alpha_1)f(z,\zeta)} \prec\prec q(z,\zeta).$$

Proof. Suppose that

$$F(z,\zeta) = \frac{H_p^{l,m}(\alpha_1+1)f(z,\zeta)}{H_p^{l,m}(\alpha_1)f(z,\zeta)}, \qquad z \in U, \ \zeta \in \overline{U}. \tag{2.8}$$

Then the function $F(z,\zeta)$ is analytic in $U \times \overline{U}$ and $F(0,\zeta) = 1$.

Differentiating both sides of (2.8) with respect to z and using (2.7), we have

$$F(z,\zeta) + zF'_z(z,\zeta)$$

$$= \frac{H_p^{l,m}(\alpha_1+1)f(z,\zeta)}{H_p^{l,m}(\alpha_1)f(z,\zeta)}$$

$$+ \frac{H_p^{l,m}(\alpha_1)f(z,\zeta)(H_p^{l,m}(\alpha_1+1)f(z,\zeta))'_z - H_p^{l,m}(\alpha_1+1)f(z,\zeta)(H_p^{l,m}(\alpha_1)f(z,\zeta))'_z}{[H_p^{l,m}(\alpha_1)f(z,\zeta)]^2}$$

$$= \frac{H_p^{l,m}(\alpha_1)f(z,\zeta)(zH_p^{l,m}(\alpha_1+1)f(z,\zeta))'_z - zH_p^{l,m}(\alpha_1+1)f(z,\zeta)(H_p^{l,m}(\alpha_1)f(z,\zeta))'_z}{[H_p^{l,m}(\alpha_1)f(z,\zeta)]^2}$$

$$= \left(\frac{zH_p^{l,m}(\alpha_1+1)f(z,\zeta)}{H_p^{l,m}(\alpha_1)f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta).$$

An application of Lemma 1.2, we obtain

$$\frac{H_p^{l,m}(\alpha_1+1)f(z,\zeta)}{H_p^{l,m}(\alpha_1)f(z,\zeta)} \prec\prec q(z,\zeta).$$

Theorem 2.4. *Let* $q(z,\zeta)$ *be a convex function such that* $q(0,\zeta) = 1$ *and let* h *be the function* $h(z,\zeta) = q(z,\zeta) + \frac{1}{\alpha_1 + p} zq'_z(z,\zeta)$, *where* $\alpha_1 > 0$. *Suppose that*

$$G(z, \zeta) = \frac{\alpha_1 + p}{z^{\alpha_1}} \int_0^z t^{\alpha_1 - 1} f(t, \zeta) dt, \quad z \in U, \ \zeta \in \overline{U}. \tag{2.9}$$

If $f \in \mathcal{A}_\zeta^*(p)$ *satisfies the strong differential subordination*

$$\frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))_z'}{pz^{p-1}} \prec\prec h(z, \zeta), \tag{2.10}$$

then

$$\frac{(H_p^{l,m}(\alpha_1) G(z, \zeta))_z'}{pz^{p-1}} \prec\prec q(z, \zeta).$$

Proof. Suppose that

$$F(z, \zeta) = \frac{(H_p^{l,m}(\alpha_1) G(z, \zeta))_z'}{pz^{p-1}}, \quad z \in U, \ \zeta \in \overline{U}. \tag{2.11}$$

Then the function $F(z, \zeta)$ is analytic in $U \times \overline{U}$ and $F(0, \zeta) = 1$.

From (2.9), we have

$$z^{\alpha_1} G(z, \zeta) = (\alpha_1 + p) \int_0^z t^{\alpha_1 - 1} f(t, \zeta) dt. \tag{2.12}$$

Differentiating both sides of (2.12) with respect to z, we get

$$(\alpha_1 + p) f(z, \zeta) = \alpha_1 G(z, \zeta) + zG_z'(z, \zeta)$$

and

$$(\alpha_1 + p) H_p^{l,m}(\alpha_1) f(z, \zeta) = \alpha_1 H_p^{l,m}(\alpha_1) G(z, \zeta) + z(H_p^{l,m}(\alpha_1) G(z, \zeta))_z'.$$

So

$$\frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))_z'}{pz^{p-1}} = \frac{(\alpha_1 + 1)}{p(\alpha_1 + p)} \frac{(H_p^{l,m}(\alpha_1) G(z, \zeta))_z'}{z^{p-1}}$$

$$+ \frac{1}{p(\alpha_1 + p)} \frac{(H_p^{l,m}(\alpha_1) G(z, \zeta))_{z^2}''}{z^{p-2}}. \tag{2.13}$$

From (2.11) and (2.13), we obtain

$$F(z, \zeta) + \frac{1}{\alpha_1 + p} zF_z'(z, \zeta) = \frac{(H_p^{l,m}(\alpha_1) f(z, \zeta))_z'}{pz^{p-1}}. \tag{2.14}$$

Using (2.14), (2.10) becomes

$$F(z, \zeta) + \frac{1}{\alpha + 2p} zF_z'(z, \zeta) \prec\prec q(z, \zeta) + \frac{1}{\alpha + 2p} zq_z'(z, \zeta).$$

An application of Lemma 1.2 yields $F(z, \zeta) \prec\prec q(z, \zeta)$. By using (2.10), we obtain.

$$\frac{(H_p^{l,m}(\alpha_1) G(z, \zeta))_z'}{pz^{p-1}} \prec\prec q(z, \zeta).$$

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