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New Iterative Methods and Sensitivity Analysis for Inverse Quasi Variational Inequalities

(Dedicated to our respected, beloved and visionary parents)

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Abstract

Some classes of inverse quasi variational inequalities, which can be viewed as a novel important special case of quasi variational equalities, introduced in Noor [47] in 1988, are investigated. Using various techniques such as Wiener-Hopf equations, auxiliary principle, dynamical systems coupled with finite difference approach we suggest and analyzed a number of new and known numerical techniques for solving inverse quasi variational inequalities. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis is also investigated. One can obtain a number of new classes of inverse variational inequalities by interchanging the role of operators. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

1 Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory was introduced and considered in early sixties by Lions and Stampacchia [35], can be viewed as a novel extension and generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using novel and innovative ideas to handle complicated and complex problems. Noor [46,47] considered two new classes of variational inequalities involving two arbitrary operators in 1988, which are known as general variational inequalities

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and have applications in oceanography, non-positive and non symmetric differential equations theory. An important special case of these general variational inequalities is known as inverse variational inequalities.

It has been established that the variational inequalities are equivalent to the fixed point problems This equivalent formulations have played an important role to study the existence of the solution and to develop efficient numerical methods for solving variational inequalities and related optimization problems. Noor [51, 54] has proposed and suggested three step forward-backward iterative methods, known as Noor iterations, for finding the approximate solution of general variational inequalities using the technique of updating the solution and auxiliary principle. These forward-backward splitting algorithms are similar to those of the schemes of Glowinski and Le Tallec [21], which they suggested by using the Lagrangian technique. Suantai et al. [83] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image inpainting. Ashish et al. [4–6], Cho et al. [12] and Kwuni et al. [34] explored the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations. It is worth mentioning that Noor iterations have influenced the research in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. These three-step schemes are a natural generalization of the splitting methods of Ames [3] for solving partial differential equations. Noor (three-step) iterations contain Mann (one-step) iteration and Ishikawa (two-step) iterations as special cases. Inspired and motivated by the usefulness and applications of the splitting three-step methods, several classes of three-step approximation schemes for solving variational inequalities, fixed points and related problems are being investigated. It has been established [51, 79] that Noor (Three step) iterations, perform better than the Ishikawa (two step) iterations and one step method Mann (one step) iteration.

In various cases, the underlying set may depends upon the solution explicitly or implicity, then the variational inequality is called the quasi-variational inequality. Bensoussan and Lions [9] studied such type of problems in the field of impulse control.

To be more precise, for given nonlinear operators $\mathcal{T}, g : \mathcal{H} \leftarrow \mathcal{H}$ and convex valued closed convex set $\Omega(\mu)$, find $\mu \in \Omega(\mu) \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{1.1}$$

which is called the general quasi variational inequality considered by Noor [47] in 1988.

For $\mathcal{T} = I$, identity operator, the problem (1.1) reduces to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mu, \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{1.2}$$

is called the inverse quasi variational inequality, which have been used to consider a road pricing problem in which the environment impact problem due to traffic flow is taken into account to fix the road taxes and a bipartite market equilibrium problems. Also, a number of problems such as flow control problems, transportation, telecommunication networks, dynamic power price problem and least distance problem [10] can be studied in the unified framework of inverse variational inequalities. Ironically, all the authors and researchers in [6, 22–27, 84, 86] have not cited the original paper on quasi variational inequalities by Noor [47], which is unethical. They have copied and pasted all the results from various research articles of Noor and his coauthors with minor modifications.

Noor [43, 47] proved that the quasi variational inequalities are equivalent to the implicit fixed point problem. This equivalent formulation played an important role in developing numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi-variational inequalities. For the applications, motivations, generalizations, extensions, dynamical systems, sensitivity analysis, numerical methods, error bounds and related optimization programming problems, see [1, 2, 9, 11, 13, 17–21, 28–31, 33, 34, 37, 38, 41–43, 45–75, 77, 83, 85, 90] and the references therein.

The Wiener-Hopf equations were introduced and studied by Shi [80] and Robinson [78]. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. Noor [50] have proved that quasi variational inequalities are equivalent to the Wiener-Hopf equations. This equivalence has been used to study the existence and stability of the solution of quasi variational inequalities.

The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [19]. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. This dynamical system is a first order initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown [17, 19, 31, 38, 54, 64, 69, 87, 88] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems.

We would like to mention that the sensitivity analysis provides useful information for designing or planning various equilibrium systems. Sensitivity analysis can provide new insight and stimulate new ideas and techniques for problem solving. Dafermos [17] studied the sensitivity analysis of the variational inequalities using the fixed point technique. This approach has strong geometrical flavour and has been investigated for various classes of quasi variational inequalities. Also see, [50, 54, 62, 63, 73, 84, 86] and the references therein.

Motivated and inspired by ongoing recent research in variational inequalities, we revisit the inverse quasi variational inequalities, which is a special case of quasi variational inequalities involving two arbitrary operators, introduced and studied by Noor [47] in 1988. Noor [43] established the equivalence between the quasi variational inequalities and fixed point problem, which has been used to consider an iterative method for solving quasi variational inequalities. We prove that the nonlinear programming problems and implicit second order obstacle boundary value problems can be studied via the inverse quasi variational inequalities. Several special cases are discussed as applications of the inverse quasi variational inequalities, discussed in Section 2. In section 3, we discuss the unique existence of the solution as well as to suggest several inertial iterative method along with the convergence analysis. The inverse Wiener-Hopf equation technique is used to suggest some iterative methods in Section 4. We also apply the auxiliary principle technique involving an arbitrary operator is used to discuss some iterative schemes for solving the inverse quasi variational inequalities in Section 5. Dynamical system approach is applied to study the stability of the solution and to suggest some iterative methods for solving the inverse quasi variational inequalities exploring the finite difference idea. Our results in this section can be viewed as significant refinement of the results in [7, 18, 84, 86] and the references therein.

Sensitivity analysis for variational inequalities has been studied by many authors using quite different techniques. The techniques suggested so far vary with the problem being studied. Dafermos [17] used the equivalence between the variational inequalities and the fixed-point problem to study the sensitivity analysis of the classical variational inequalities. This technique has been modified and extended by many authors for studying the sensitivity analysis of various classes of quasi variational inequalities in [50, 62, 63, 73, 75]. In Section 7, we obtain some new results for the sensitivity analysis of the inverse quasi variational inequalities.

One of the main purposes of this paper is to demonstrate the close connection among various classes of algorithms for the solution of the inverse quasi variational inequalities and to point out that researchers in different field of variational inequalities and optimization have been considering parallel paths. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications of inverse quasi variational inequalities in other areas of sciences such as machine learning, artificial intelligence, data analysis, fuzzy systems, random stochastic, financial analysis and related other optimization problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Formulations and Basic Facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis [14, 15, 40], which are needed in the derivation of the main results.

Definition 2.1. *A set Ω in $\mathcal{H}$ is said to be a convex set, if*

$$\mu + \lambda(\nu - \mu) \in \Omega, \qquad \forall \mu, \nu \in \Omega, \lambda \in [0,1].$$

Definition 2.2. *A function Φ is said to be a convex function, if*

$$\Phi((1-\lambda)\mu + \lambda\nu) \le (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu), \quad \forall \mu, \nu \in \Omega, \quad \lambda \in [0,1].$$

Convex functions are closely related to the integral inequalities and variational inequalities. These type of inequalities have played crucial part in developing fields such as: numerical analysis, operations research, transportation, financial mathematics, structural analysis, dynamical systems, sensitivity analysis and machine learning.

If the convex function Φ is differentiable, then $\mu \in \Omega$ is the minimum of the function Φ, if and only if, $\mu \in \Omega$ satisfies the inequality

$$\langle \Phi'(\mu), \nu - \mu \rangle \ge 0, \quad \forall \nu \in \Omega. \tag{2.1}$$

The inequalities of the type (2.1) are called the variational inequalities, which were introduced and studied by Lions and Stampacchia [35]. It is known that the problem (2.1) occurs, which may not be derivative of the differentiable functions. These facts and observations motivated Lions and Stampacchia [35] to consider more general variational inequalities of which (2.1) is a special case. To be more precise, for a given nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \ge 0, \quad \forall \nu \in \Omega. \tag{2.2}$$

which is called the variational inequality. Note that, for $\Phi'(\mu) = \mathcal{T}\mu$, problem (2.2) is exactly the problem (2.1).

For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [1, 9, 13, 17–21, 28–31, 33, 34, 37, 38, 41–43, 45–56, 58–66, 68–75, 77, 83, 85] and the references therein.

In many cases, the set and function may not be a convex set and convex functions. To overcome these drawbacks, Noor [57, 58] introduced the concept of general new convex set and general convex function with respect to an arbitrary function g. For the sake of completeness and to convey an idea of this result, we include some details.

Definition 2.3. *[57, 58] A set $\Omega_g \subseteq \mathcal{H}$ is said to be a general convex set, if there exists an arbitrary function $g : \mathcal{H} \longrightarrow \mathcal{H}$ such that*

$$g(\mu) + t(\nu - g(\mu)) \in \Omega_g, \quad \forall \mu, \nu \in \Omega_g, \quad t \in [0,1].$$

Note that every convex set is general convex set, but the converse is not true, see Noor [57, 58]. It is worth mentioning that the general convex (g-convex) set is different than the E-convex set of Youness [89] and various general convex set. For the applications of the general convex sets in information technology, railway systems, computer aided design, digital vector optimization and comparison with other concepts, see [14–16, 57]. If $g = I$, then the general convex set Ω_g is exactly the convex set Ω.

Definition 2.4. *The function $\Phi : \Omega_g \longrightarrow \mathcal{H}$ is said to be general convex, if there exists an arbitrary function g, such that*

$$\Phi(g(\mu) + t(\nu - g(\mu))) \leq \Phi(g(\mu)) + t\{\Phi(\nu - \Phi(g(\mu))\}, \quad \forall \mu, \nu \in \Omega_g, \quad t \in [0, 1].$$

Clearly every convex function is a general convex, but the converse is not true. For the differentiable general convex function, we have

Theorem 2.1. *[57, 58] Let Φ be a differentiable general convex function on the general convex set Ω_g. Then the minimum $\mu \in \Omega_g$ of the function Φ, if and only if, $\mu \in \Omega_g$ satisfies the inequality*

$$\langle \Phi'(g(\mu), \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega_g, \tag{2.3}$$

where $\Phi'(.)$ is the differential of Φ at $\mu \in \Omega_g$ in the direction $\nu - g(\mu)$.

Theorem 2.1 implies that general convex programming problems can be studied via the general variational inequality(2.3).

It is known that the inequality of the type (2.3) may not arise as the optimality condition of the differentiable functions.

Noor [58] introduced and investigated the problem of finding $\mu \in \Omega \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.4}$$

which is called the general variational inequalities. For the applications, motivations, numerical results, dynamical systems and related optimizations, see [60, 75]. It is worth mentioning that the problem (2.3 is a special case of the of the problem (2.4).

In many applications, the convex set Ω depends upon the solution explicitly or implicitly. In such cases, variational inequality is called the quasi variational inequality. Let $\Omega : \mathcal{H} \longrightarrow \mathcal{H}$ be a set-valued mapping which, for any element $\mu \in \mathcal{H}$, associates a convex-valued and closed set $\Omega(\mu) \subseteq \mathcal{H}$. We now consider some new classes of general quasi variational inequalities, which include several new and known classes of variational inequalities as special cases.

For given nonlinear operators $\mathcal{T}, g$, we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.5}$$

which is called the quasi variational inequality, introduced and studied by Noor [47] in 1988. Also by interchanging the role of the operators $\mathcal{T}$ and g, the problem (2.5) is equivalent to finding $\mu \in \Omega(\mu)$, such that

$$\langle g(\mu), \nu - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.6}$$

Note the symmetry role played by the mappings $\mathcal{T}$ and g. It is clear all the results, which hold for the problem (2.5), continue to hold for the problem (2.6) and vice versa.

If $\mathcal{T} = I$, the identity operator, then the problem (2.6) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle \mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.7}$$

is called the inverse quasi variational inequality, see [7, 10, 22–27, 84, 86] and the references therein.

For $g = I$, the problem (2.6) reduces to finding $\nu \in \Omega(\mu)$, such that

$$\langle \mu, \nu - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.8}$$

which is also called the inverse quasi variational inequality. Consequently, it is evident that all the known results for quasi variational inequalities are also valid for both types of inverse quasi variational inequalities. This is a surprising fascinating fact.

Special Cases

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2.7).

(I). This problem (2.5) can be viewed as a problem of finding the minimum of general convex function [57]. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.

(II). If $\Omega^*(\mu) = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathcal{H}$, then problem (2.7) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega(\mu), \quad \mu \in \Omega^*(\mu) \quad \text{and} \quad \langle \mu, g(\mu)\rangle = 0, \tag{2.9}$$

which is known as the inverse quasi complementarity problems and appears to be a new one.

For $\Omega(\mu) = \Omega$, the convex set, the problem (2.9) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \mu \in \Omega^* \quad \text{and} \quad \langle \mu, g(\mu)\rangle = 0, \tag{2.10}$$

is called the inverse nonlinear complementarity problem [46]. Obviously inverse quasi complementarity problems include the inverse complementarity problems and linear complementarity problems. The complementarity problems were introduced and studied by Cottle et al. [13], Lemke [36], Noor [46,48,54] and Noor et al. [61,66,73,76] in game theory, management sciences and quadratic programming as special cases. This inter relations among these problems have played a major role in developing numerical results for these problems and their applications.

(III). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$, then problem (2.7) reduces to finding $\mu \in \Omega$ such that

$$\langle \mu, \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.11}$$

is known as the inverse variational inequality, which is mainly due to Noor [47] by taking $\mathcal{T} = I$, the identity operator. Noor et al. [67] have considered some multi step iterative methods for solving inverse variational inequalities (2.11).

Remark 2.1. *It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, g$, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of inverse variational inequalities, inverse complementarity problems and optimization problems as special cases of the inverse quasi variational inequalities (2.7). This shows that the problem (2.5) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the general quasi-variational inequalities and their variants.*

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the inverse quasi variational inequalities and the fixed point problems. This result is used in the analyzing the convergence analysis of the implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.1. *[47] Let $\Omega(\mu)$ be a closed and convex-valued set in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega(\mu)$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.12}$$

if and only if,

$$\mu = \Pi_{\Omega(\mu)}(z),$$

where $\Pi_{\Omega(\mu)}$ is implicit projection of $\mathcal{H}$ onto the closed convex-valued set $\Omega(\mu)$.

It is well known that the implicit projection operator $\Pi_{\Omega(\mu)}$ is not nonexpansive, but it is required to satisfy the following assumption, which plays an important part in the derivation of the results..

Assumption 2.1.

$$\|\Pi_{\Omega(\mu)}\omega - \Pi_{\Omega(\nu)}\omega\| \leq \eta\|\mu - \nu\|, \forall \mu, \nu, \omega \in \mathcal{H}, \tag{2.13}$$

where $\eta > 0$ is a constant.

Assumption 2.1 has been used to prove the existence of a solution of general quasi variational inequalities as well as in analyzing convergence of the iterative methods.

In many important applications, the convex-valued set $\Omega(\mu)$ can be written as

$$\Omega(\mu) = m(\mu) + \Omega,$$

is known as the moving convex set,where $m(\mu)$ is a point-point mapping and Ω is a convex set. In this case, we have

$$\Pi_{\Omega(\mu)}\omega = \Pi_{m(\mu)+\Omega} = m(\mu) + \Pi_{\Omega}[w - m(\mu)], \quad \forall \mu, w \in \Omega.$$

We note that, if $m(\mu)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then

$$\begin{aligned} \|\Pi_{\Omega(\mu)}w - \Pi_{\Omega(\nu)}w\| &= \|m(\mu) - m(\nu) + \Pi_{\Omega}[w - m(\mu)] - \Pi_{\Omega}[w - m(\nu)\| \\ &\leq 2\|m(\mu) - m(\nu)\| \leq 2\gamma\|\mu - \nu\|, \quad \forall \mu, \nu, w \in \Omega. \end{aligned}$$

which shows that Assumption 2.1 holds with $\eta = 2\gamma$.

Definition 2.5. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:*

1. *Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

2. *Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$\|\mathcal{T}\mu - \mathcal{T}\nu\| \leq \beta\|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

3. *Monotone, if*

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

4. *Pseudo monotone, if*

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}\nu, \nu - \mu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.2. *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

3 Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the inverse quasi variational inequalities.

Using Lemma 2.1, one can show that the inverse quasi variational inequalities are equivalent to the fixed point problems.

Lemma 3.1. *The function $\mu \in \Omega(\mu)$ is a solution of the inverse quasi variational inequality (2.7), if and only if, $\mu \in \Omega(\mu)$ satisfies the relation*

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu], \tag{3.1}$$

where $\Pi_{\Omega(\mu)}$ is the implicit projection operator and $\rho > 0$ is a constant.

Proof. Let $u \in \Omega(\mu)$ be a solution of the problem (2.7). Then

$$\langle \rho\mu + g(\mu) - g(\mu), \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu).$$

Using Lemma 2.1, we have

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu],$$

the required result. □

Lemma 3.1 implies that the inverse quasi variational inequality (2.7) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation (3.1) will play an important role in deriving the main results.

From the equation (3.1), we have

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu].$$

We define the function F associated with (3.1) as

$$F(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu], \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.7), it is enough to show that the map F defined by (3.2) has a fixed point.

Theorem 3.1. *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, respectively. If the Assumption 2.1 holds and there exists a parameter $\rho > 0$, such that*

$$\rho < 1 - k, \quad k < 1, \tag{3.3}$$

where

$$\theta = \rho + k \tag{3.4}$$
$$k = \sqrt{1-2\sigma+\zeta^2} + \eta + \zeta. \tag{3.5}$$

then there exists a unique solution of the problem (2.5).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.7) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2) has a fixed point.

For all $\nu \neq \mu \in \Omega(\mu)$, we have

$$\begin{aligned}
\|F(\mu)-F(\nu)\| &= \|\mu-\nu-(g(\mu)-g(\nu))\| + \Pi_{\Omega(\mu)}\|[g(\mu)-\rho\mu] - \Pi_{\Omega(\nu)}[g(\nu)-\rho\nu]\| \\
&= \|\nu-\mu-(g(\nu)-g(\mu))\| + \|\Pi_{\Omega(\mu)}[g(\nu)-\rho\nu] - \Pi_{\Omega(\nu)}[g(\nu)-\rho\nu]\| \\
&+ \|\Pi_{\Omega(\nu)}[g(\nu)-\rho\nu] - \Pi_{\Omega(\nu)}[g(\mu)-\rho\mu]\| \\
&\leq \|\mu-\nu-(g(\mu)-g(\nu))\| + \eta\|\nu-\mu\| + \|g(\nu)-g(\mu)-\rho(\nu-\mu)\| \\
&\leq \|\mu-\nu-(g(\mu)-g(\nu)\| + \eta\|\nu-\mu\| + \zeta\|\nu-\mu\| + \rho\|\nu-\mu\|.
\end{aligned} \tag{3.6}$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu-\nu-(g(\mu)-g(\nu))\|^2 &\leq \|\mu-\nu\|^2 - 2\langle g(\mu)-g(\nu), \mu-\nu\rangle + \zeta^2\|g(\mu)-g(\nu)\|^2 \\
&\leq (1-2\sigma+\zeta^2)\|\mu-\nu\|^2.
\end{aligned} \tag{3.7}$$

From (4.5) and (3.7), we have

$$\begin{aligned}
\|F(\mu)-F(\nu)\| &\leq 2\left\{\sqrt{(1-2\sigma+\zeta^2)} + \eta + \zeta + \rho\right\}\|\mu-\nu\| \\
&= \theta\|\mu-\nu\|,
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \rho + k \\
k &= 2\sqrt{1-2\sigma+\zeta^2} + \eta + \zeta.
\end{aligned}$$

From (3.3), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (3.2) has a fixed point, which is the unique solution of (2.7). □

The fixed point formulation (3.1) is applied to propose and suggest the iterative methods for solving the problem (2.7).

This alternative equivalent formulation (3.1) is used to suggest the following iterative methods for solving the problem (2.7).

Algorithm 3.1. *For a given μ_0, compute the approximate solutions $\{\mu_n\}$, $\{w_n\}$ and $\{y_n\}$ by the iterative schemes*

$$\begin{aligned} g(y_n) &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n] \\ g(w_n) &= \Pi_{\Omega(y_n)}[g(y_n) - \rho y_n] \\ g(\mu_{n+1}) &= \Pi_{\Omega(w_n)}[g(w_n) - \rho w_n]. \end{aligned}$$

Algorithm 3.1 is a three step forward-backward splitting algorithm for solving inverse quasi variational inequality (2.7). This method is very much similar to that of Glowinski et al. [21] for variational inequalities, which they suggested by using the Lagrangian technique.

We now suggested and analyzed the three step iterative methods for solving the inverse quasi variational inequality (2.7).

Algorithm 3.2. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$y_n = (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n]\} \tag{3.8}$$

$$w_n = (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho y_n]\} \tag{3.9}$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_{\Omega(w_n)}[w_n - \rho w_n]\}. \tag{3.10}$$

which are known as Noor iterations.

For $\gamma_n = 0$, Algorithm 3.2 reduces to:

Algorithm 3.3. *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} w_n &= (1-\beta_n)\mu_n + \beta_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n]\} \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_{\Omega(\mu_n)}[g(w_n) - \rho w_n\}, \end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (2.7).

Note that for $\gamma_n = 0$ *and* $\beta_n = 0$, Algorithm 3.1 is called the Mann iterative method, that is.

Algorithm 3.4. *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\mu_{n+1} = (1-\beta_n)\mu_n + \beta_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n]\}.$$

We suggest new perturbed iterative schemes for solving the inverse quasi variational inequality (2.7).

Algorithm 3.5. *For a given μ_0, compute the approximate solution $\{\mu_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n]\} + \gamma_n h_n \\ w_n &= (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho y_n]\} + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_{\Omega(w_n)}[g(w_n) - \rho w_n\} + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and $\Pi_{\Omega(\mu_n)}$ is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \le \alpha_n, \beta_n, \gamma_n \le 1; \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving inverse quasi variational inequality (2.7).

We now study the convergence analysis of Algorithm 3.2, which is the main motivation of our next result.

Theorem 3.2. *Let the operator g satisfy all the assumptions of Theorem 3.1. If the condition (3.3) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3.2 converges to the exact solution $\mu \in \Omega(\mu)$ of the inverse quasi variational inequality (2.7) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu \in \Omega(\mu)$ of the inverse quasi variational inequalities (2.7). Let $\mu \in \Omega(\mu)$ be the unique solution of (2.7). Then, using Lemma 3.1, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu]\} && (3.11) \\ &= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu]\} && (3.12) \\ &= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu]\}. && (3.13) \end{aligned}$$

From (3.10), (3.11) and Assumption (2.1), we have

$$\begin{aligned}
\|\mu_{n+1}-\mu\| &= \|(1-\alpha_n)(\mu_n-\mu)+\alpha_n(w_n-\mu-(g(w_n)-g(\mu))) \\
&+ \alpha_n\Pi_{\Omega((w_n)}[g(w_n)-\rho w_n]-\Pi_{(\mu)}[g(\mu)-\rho\mu\}\| \\
&\le (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\|w_n-\mu-(g(w_n)-g(\mu))\| \\
&+ \alpha_n\Pi_{\Omega((w_n)}[g(w_n)-\rho w_n]-\Pi_{\Omega(w_n)}[g(\mu_n)-\rho\mu\}\| \\
&+ \alpha_n\{\Pi_{(w_n)}[g(\mu_n)-\rho\mu]-\Pi_{\Omega(\mu)}[g(\mu)-\rho\mu\}\| \\
&\le (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\|w_n-\mu-(g(w_n)-g(\mu))\| \\
&+ \alpha_n\|g(w_n)-g(\mu)-\rho(w_n-\mu)||+\alpha_n\eta\|w_n-\mu\| \\
&\le (1-\alpha_n)\|\mu_n-\mu||+\alpha_n(k+\rho)||w_n-\mu\| \\
&= (1-\alpha_n)||u_n-\mu\|+\alpha_n\theta||w_n-\mu\|, \qquad (3.14)
\end{aligned}$$

where θ is defined by (3.4).

In a similar way, from (3.8) and (3.12), we have

$$\begin{aligned}
\|w_n-\mu\| &\le (1-\beta_n)\|\mu_n-\mu\|+2\beta_n\theta\|y_n-\mu-(g(y_n)-g(\mu))\| \\
&+ \beta_n\|g(y_n)-g(\mu)-\rho(y_n-\mu)\|+\beta_n\eta\|y_n-\mu\| \\
&\le (1-\beta_n)\|\mu_n-\mu\|+\beta_n(k+\rho)\|y_n-\mu\|, \\
&\le (1-\beta_n)\|\mu_n-\mu||+\beta_n\theta\|y_n-\mu\|, \qquad (3.15)
\end{aligned}$$

where θ is defined by (3.3).

From (3.8) and (3.13), we obtain

$$\begin{aligned}
\|y_n-\mu\| &\le (1-\gamma_n)\|\mu_n-\mu\|+\gamma_n\theta\|\mu_n-\mu\| \\
&\le (1-(1-\theta)\gamma_n)\|\mu_n-\mu\| \\
&\le ||\mu_n-\mu||. \qquad (3.16)
\end{aligned}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned}
\|w_n-\mu\| &\le (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|\mu_n-\mu\| \\
&= (1-(1-\theta)\beta_n)\|\mu_n-\mu\| \\
&\le ||\mu_n-\mu||. \qquad (3.17)
\end{aligned}$$

Form the above equations, we have

$$\begin{aligned}
\|\mu_{n+1}-\mu\| &\le (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\theta\|\mu_n-\mu\| \\
&= [1-(1-\theta)\alpha_n]\|\mu_n-\mu\| \\
&\le \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0-\mu\|.
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (3.16), and (3.17), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

Also, we can suggest the following iterative methods for solving the inverse quasi variational inequalities.

Algorithm 3.6. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mu_n],\quad n=0,1,2,... \tag{3.18}$$

which is known as the projection method.

Algorithm 3.7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_{n+1})}[g(\mu_n)-\rho\mu_{n+1}],\quad n=0,1,2,... \tag{3.19}$$

which is known as the implicit projection method and is equivalent to the following two-step method.

Algorithm 3.8. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mu_n]\\ \mu_{n+1} &= \mu_n-g(\mu_n)+\Pi_{\Omega(\omega_n)}[g(\mu_n)-\rho\omega_n],\quad n=0,1,2,...\end{aligned}$$

Algorithm 3.9. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1})-\rho\mu_{n+1}],\quad n=0,1,2,... \tag{3.20}$$

which is known as the modified projection method and is equivalent to the iterative method.

Algorithm 3.10. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho u_n]\\ \mu_{n+1} &= \mu_n-g(\mu_n)+\Pi_{\Omega(\omega_n)}[g(\omega_n)-\rho\omega_n],\quad n=0,1,2,...\end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.7).

We can rewrite the equation (3.1) as:

$$\mu=\mu-g(\mu)+\Pi_{\Omega(\mu)}[g(\frac{\mu+\mu}{2})-\rho\mu]. \tag{3.21}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 3.11. *[46]. For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\mu_{n+1}]. \tag{3.22}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (2.7) .

Algorithm 3.12. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g(\frac{\omega_n + \mu_n}{2}) - \rho\omega_n]. \end{aligned}$$

One can rewrite (3.1) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[(\frac{\mu + \mu}{2}) - \rho\mathcal{T}(\frac{\mu + \mu}{2})]. \tag{3.23}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.5).

Algorithm 3.13. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho(\frac{\mu_n + \mu_{n+1}}{2})]. \tag{3.24}$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.9 as the predictor and Algorithm 3.13 as corrector. Thus, we obtain a new two-step method for solving the problem (2.7).

Algorithm 3.14. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\left[g(\left(\frac{\omega_n + \mu_n}{2}\right) - \rho\left(\frac{\omega_n + \mu_n}{2}\right)\right], \end{aligned}$$

which is a new predictor-corrector two-step method.

For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g((1-\xi)\mu + \xi\mu) - \rho((1-\xi)\mu + \xi\mu)].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.5).

Algorithm 3.15. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g((1-\xi)\mu_n + \xi\mu_{n-1}) - \rho((1-\xi)\mu_n + \xi\mu_{n-1})], \quad n = 0, 1, 2, ...$$

It is noted that Algorithm 3.15 is equivalent to the following two-step method.

Algorithm 3.16. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\omega_n) - \rho\omega_n]. \end{aligned}$$

Algorithm 3.16 is known as the inertial projection method.

We now suggest multi-step inertial methods for solving the inverse quasi variational inequalities (2.7).

Algorithm 3.17. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ y_n &= (1-\beta_n)\omega_n + \beta_n\left\{\omega_n - g(\omega_n) + \Pi_{(\omega_n)}\left[g(\frac{\omega_n + \mu_n}{2}) - \rho(\frac{\omega_n + \mu_n}{2})\right]\right\}, \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n\left\{y_n - g(y_n) + \Pi_{\Omega(\mathtt{y}_n)}\left[g(\frac{\omega_n + \mathtt{y}_n}{2}) - \rho(\frac{\mathtt{y}_n + \omega_n}{2})\right]\right\}, \end{aligned}$$

where $\Theta_n \in [0, 1], \forall n \geq 1.$

Algorithm 3.17 is a three-step modified inertial method for solving inverse quasi variational inequalities (2.7).

Similarly a four-step inertial method for solving the inverse quasi variational inequalities (2.7) is suggested.

Algorithm 3.18. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}), \\ y_n &= (1-\gamma_n)\omega_n + \gamma_n\left\{\omega_n - g(\omega_n) + \Pi_{(\omega_n)}\left[g(\frac{\omega_n + \mu_n}{2}) - \rho(\frac{\omega_n + \mu_n}{2})\right]\right\}, \\ z_n &= (1-\beta_n)y_n + \beta_n\left\{y_n - g(y_n) + \Pi_{\Omega(y_n)}\left[g(\frac{y_n + \omega_n + \mu_n}{3}) - \rho(\frac{y_n + \omega_n + \mu_n}{3})\right]\right\}, \\ \mu_{n+1} &= (1-\alpha_n)z_n + \alpha_n\left\{z_n - g(z_n) + \Pi_{\Omega(z_n)}\left[g(\frac{z_n + y_n + \omega_n}{3}) - \rho(\frac{y_n + z_n + \omega_n}{3})\right]\right\}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0, 1], \quad \forall n \geq 1.$

Using the technique of Noor et al. [71] and Jabeen et al. [28–30], one can investigate the convergence analysis of these inertial projection methods. We would like to mention that Algorithm 3.17 and Algorithm 3.18 can be viewed as the generalizations of Noor (three-step) iterations [51,54,63] for solving the inverse quasi variational inequalities. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving system of inverse quasi variational inequalities, which is an interesting problem.

4 Wiener-Hopf Equations Technique

In this section, we discuss the Wiener-Hopf equations associated with the quasi variational inequalities. It is worth mentioning that the Wiener-Hopf equations associated with variational inequalities were introduced and studied by Shi [80] and Ronbinson [78] independently using different techniques. Noor [50] proved that the quasi variational inequalities are equivalent to the implicit Wiener-Hopf equations.

We now consider the problem of solving the inverse Wiener-Hopf equations related to the inverse quasi variational inequalities. Let $\mathcal{R}_{\Omega(\mu)} = \mathcal{I} - \Pi_{\Omega(\mu)}$, and $\Pi_{\Omega(\mu)}$ be the projection operator.

We consider the problem of finding $z \in \mathcal{H}$ such that

$$\Pi_{\Omega(\mu)} z + \rho^{-1} \mathcal{R}_{\Omega(\mu)} z = 0. \tag{4.1}$$

The equations of the type (4.1) are called the inverse Wiener-Hopf equations. We apply the inverse Wiener-Hopf equations to consider some iterative methods, sensitivity analysis and other aspects of the inverse quasi variational inequalities.

Lemma 4.1. *The element $\mu \in \Omega(\mu)$ is a solution of the inverse quasi variational inequality (2.7), if and only if, $z \in \mathcal{H}$ satisfies the resolvent equation (4.1), where*

$$g(\mu) = \Pi_{\Omega(\mu)} z, \tag{4.2}$$

$$z = g(\mu) - \rho\mu, \tag{4.3}$$

where $\rho > 0$ is a constant.

From Lemma 4.1, it follows that the inverse quasi variational inequalities (2.7) and the inverse Wiener-Hopf equations (4.1) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the inverse quasi variational inequalities and related optimization problems.

We use the inverse Wiener-Hopf equations (4.1) to suggest some new iterative methods for solving the inverse quasi variational inequalities. From (4.2) and (4.3),

$$\begin{aligned} z &= \Pi_{\Omega(\mu)}z - \rho g^{-1}\Pi_{\Omega(\mu)}z \\ &= g^{-1}\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \rho g^{-1}\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu]. \end{aligned}$$

Thus, we have

$$g(\mu) = \Pi_{\Omega(\mu)}\Big[g^{-1}\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \rho g^{-1}\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu]\Big]$$

implies that

$$\mu = (1-\alpha_n)\mu + \alpha_n\bigg(g(\mu) - \Big[\Pi_{\Omega(\mu)}\Big[g^{-1}\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \rho g^{-1}\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu]\Big]\Big]\bigg)$$

Consequently, for a constant $\alpha_n > 0$, we can suggest the following new predictor-corrector method for solving the inverse quasi variational inequalities (2.7).

Algorithm 4.1. *For given u_0, compute u_{n+1} by the iterative scheme*

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\Big[\Big[g(\mu_n) - \Pi_{\Omega(\mu_n)}\Big[g(w_n) - \rho w_n\Big]\Big] \tag{4.4}$$

where

$$g(w_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n], \tag{4.5}$$

which appears to be a new one.

In a similar way, we can suggest and analyse the predictor-corrector method for solving the inverse quasi variational inequalities (2.7), which only involve only one projection.

Algorithm 4.2. *For given u_0, u_1, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - \xi(\mu_n - \mu_{n-1}) \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[\rho\omega_n - \rho\mu_n]. \end{aligned}$$

Remark 4.1. *We have only given some glimpse of the technique of the inverse Wiener-Hopf equations for solving the inverse quasi variational inequalities. One can explore the applications of the inverse Wiener-Hopf equations in developing efficient numerical methods for variational inequalities and related nonlinear optimization problems.*

5 Auxiliary Principle Technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al [20] as developed in [49,54,73,74,77], to suggest and analyze some proximal point methods for solving general quasi variational inequalities (2.5).

We apply the auxiliary principle technique involving an arbitrary operator, which is mainly due to Noor [49], for finding the approximate solution of the inverse quasi variational inequalities (2.7).

For a given $\mu \in \Omega(\mu)$ satisfying (2.7), find $w \in \Omega(\mu)$ such that

$$\langle \rho(w + \eta(\mu - w)), \nu - g(w)\rangle + \langle M(w) - M(\mu), \nu - w\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.1}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (8.4) is called the auxiliary inverse quasi variational inequality.

If $w = \mu$, then w is a solution of (2.7). This simple observation enables us to suggest the following iterative method for solving (2.7).

Algorithm 5.1. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1})\rangle \\ &\quad + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu). \end{aligned} \tag{5.2}$$

Algorithm 5.1 is called the hybrid proximal point algorithm for solving the inverse quasi variational inequalities (2.7).

Special Cases: We now discuss some special cases.

(I). For $\eta = 0$, Algorithm 5.1 reduces to

Algorithm 5.2. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho\mu_{n+1}, \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.3}$$

is called the implicit iterative methods for solving the problem (2.7).

(II). If $\eta = 1$, then Algorithm 5.1 collapses to

Algorithm 5.3. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho\mu_n, \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

is called the explicit iterative method.

(III). For $\eta = \frac{1}{2}$, Algorithm 5.1 becomes:

Algorithm 5.4. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho(\frac{\mu_{n+1} + \mu_n}{2}), \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

is known as the mid-point proximal method for solving the problem (2.7).

For the convergence analysis of Algorithm 5.2, we need the following concepts.

Definition 5.1. *An operator g is said to be pseudomontone, if*

$$\langle \mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

implies that

$$-\langle \nu, g(\mu) - \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu).$$

Theorem 5.1. *Let the operator g be a pseudo-monotone. Let the approximate solution μ_{n+1} obtained in Algorithm 5.2 converges to the exact solution $\mu \in \Omega(\mu)$ of the problem (2.7). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu - \mu_n\|. \tag{5.4}$$

Proof. Let $\mu \in \Omega(\mu)$ be a solution of the problem (2.7). Then

$$-\langle \rho\nu, g(\mu) - \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.5}$$

since the operator g is a pseudo-monotone.

Taking $\nu = \mu_{n+1}$ in (5.5), we obtain

$$-\langle \rho\mu_{n+1}, g(\mu) - \mu_{n+1}\rangle \geq 0. \tag{5.6}$$

Setting $\nu = \mu$ in (8.6), we have

$$\langle \rho\mu_{n+1}, g(\mu) - \mu_{n+1}\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \geq 0. \tag{5.7}$$

Combining (5.7), (5.6) and (5.5), we have

$$\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \quad \geq \quad -\langle \rho\mu_{n+1}, g(\mu) - u_{n+1}\rangle \geq 0. \tag{5.8}$$

From the equation (5.8), we have

$$\begin{aligned} 0 &\leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (5.4). □

Theorem 5.2. *Let H be a finite dimensional space and all the assumptions of Theorem 5.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 5.2 converges to the exact solution $\mu \in \Omega(\mu)$ of (2.7).*

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (2.7). From (5.4), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi\sum_{n=0}^{\infty}\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0. \tag{5.9}$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega(\mu)$. Replacing w_n by μ_{n_j} in (7.2), taking the limit $n_j \longrightarrow \infty$ and using (5.9), we have

$$\langle \rho\hat{\mu}, \nu - g(\hat{\mu})\rangle \geq 0, \qquad \forall \nu \in \Omega(\mu),$$

which implies that $\hat{u}$ solves the problem (2.7) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (\mu_n) = \hat{\mu}.$$

the required result. □

In recent years inertial type iterative methods have been applied to find the approximate solutions of variational inequalities and related optimizations. We again apply the modified auxiliary principle approach involving an arbitrary nonlinear operator to suggest some hybrid inertial proximal point schemes for solving the inverse quasi variational inequalities.

For a given $\mu \in \Omega(\mu)$ satisfying (2.5), find $w \in \Omega(\mu)$ such that

$$\begin{aligned} &\langle \rho(w + \eta(\mu - w)), \nu - g(w)\rangle \\ &\quad + \langle M(w) - M(\mu) + \alpha(\mu - \mu), \nu - w\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \end{aligned} \tag{5.10}$$

where $\rho > 0, \eta, \alpha \in [0,1]$ are constants and M is a nonlinear operator.

Clearly $w = \mu$, implies that w is a solution of (2.5). This simple observation enables us to suggest the following iterative method for solving (2.5).

Algorithm 5.5. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1})\rangle \\ &\quad + \langle M(\mu_{n+1}) - M(\mu_n) + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu). \end{aligned}$$

Algorithm 5.5 is called the hybrid proximal point algorithm for solving the inverse quasi variational inequalities (2.7). For $\alpha = 0$, Algorithm 5.5 is exactly Algorithm 5.1.

For $M = I$, Algorithm 5.5 reduces to the following method:

Algorithm 5.6. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1})\rangle \\ &\quad + \langle \mu_{n+1} - \mu_n + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu). \end{aligned}$$

Remark 5.1. *For different and suitable choice of the parameters ρ, η, α, operators g, M and convex-valued sets, one can recover new and known iterative methods for solving inverse quasi variational inequalities, inverse complementarity problems and related optimization problems. Using the technique and ideas of Theorem 5.1 and Theorem 5.2, one can analyze the convergence of Algorithm 5.5 and its special cases.*

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6 Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving the inverse quasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [19]. It is worth mentioning that the dynamical system is a first order initial value problem. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. For more details, see [7,18,19,31,53–55,63,64,69,73,87,88]. We consider some new iterative methods for solving the quasi variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - g(\mu)\}. \tag{6.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(2.7), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{6.2}$$

We now consider a dynamical system associated with the inverse quasi variational inequalities. Using the equivalent formulation (3.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - g(\mu)\}, \quad \mu(t_0) = \alpha, \tag{6.3}$$

where λ is a parameter. The system of type (6.3) is called the projection dynamical system associated with the problem (2.7). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.7) can be studied.

The equilibrium point of the dynamical system (6.13) is defined as follows.

Definition 6.1. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (6.13), if,*

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the inverse quasi variational inequality (2.7), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

This implies that $\mu \in \Omega(\mu)$ is a solution of the inverse quasi variational inequality (2.7), if and only if, $\mu \in \Omega(\mu)$ is an equilibrium point.

Definition 6.2. *[19] The dynamical system is said to converge to the solution set S^* of (6.3), if, irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{6.4}$$

where

$$dist(\mu, S^*) = inf_{\nu\in S^*}\|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (6.4) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 6.3. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \le u_1\|\mu(t_0) - \mu^*\|exp(-\eta(t - t_0)), \quad \forall t \ge t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 6.1. *(Gronwall Lemma) [19] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \le t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \ge t_0$,*

$$\hat{\mu} \le \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \le \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the projection dynamical system (6.3) converges to the unique solution of the inverse quasi variational inequality (2.7). The analysis is in the spirit of Noor [54, 55] and Xia and Wang [87, 88].

Theorem 6.1. *Let the operator $g : H \longrightarrow H$ be Lipschitz continuous with constant $\zeta > 0$. If $\lambda(\eta+2\zeta+\rho) < 1$ and Assumption 2.1 then, for each $\mu_0 \in \Omega\mu$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (6.3) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - g(\mu)\}, \quad \forall \mu \in H,$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.

$\forall \mu, \nu \in H$, we have

$$\begin{aligned}
\|G(\mu) - G(\nu)\| &\leq \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\nu]\| + \|g(\mu) - g(\nu)\|\} \\
&= \lambda\{\|g(\mu) - g(\nu)\| + \|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \Pi_{\Omega(\mu)}[g(\nu) - \rho\nu]\| \\
&\quad + \|\Pi_{\Omega(\mu)}[g(\nu) - \rho\nu] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\nu]\|\} \\
&\leq \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \|g(\mu) - g(\nu) - \rho\mu - \nu)\} \\
&\leq \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \{\|g(\mu) - g(\nu)\| + \rho\|\mu - \nu\|\} \\
&\leq \lambda\{(\eta + 2\zeta + \ \rho)\}\|\mu - \nu\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(\eta + 2\zeta + \rho)\} < 1$ and for each $\mu \in \Omega(\mu)$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (6.3), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \Omega(\mu)$,

$$\begin{aligned}
\|G(\mu)\| = \|\frac{d\mu}{dt}\| &= \lambda\|[g(\mu) - \rho\mu] - g(\mu)\| \\
&\leq \lambda\{\|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(\mu)\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho\mu\| + \|\Pi_{\Omega(\mu)}[g(\mu)] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(u)\|\} \\
&\leq \lambda\{(\rho + 2 + 2\zeta)\|u\| + \|\Pi_{\Omega(\mu)}[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\| ds \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds,
\end{aligned}$$

where $k_1 = \lambda\|\Pi_{\Omega(\mu)}[0]\|$ and $k_2 = \delta\lambda(\rho + 2 + 2\zeta)$. Hence by the Gronwall Lemma 6.1, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t - t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 6.2. *If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and $\zeta > 0$, then the dynamical system (6.3) converges globally exponentially to the unique solution of the general quasi variational inequality (2.7).*

Proof. Since the operator g is Lipschitz continuous, it follows from Theorem 6.1 that the dynamical system (6.3) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (6.3). For a given $\mu^* \in H$ satisfying (2.7), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \Omega(\mu). \tag{6.5}$$

From (6.3) and (6.5), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\langle\mu(t) - \mu^*, \frac{d\mu}{dt}\rangle \\
&= 2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mu(t)] - g(\mu(t))\rangle \\
&= 2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mu(t)] - g(\mu^*) + g(\mu^*) - g(\mu(t))\rangle \\
&= -2\lambda\langle\mu(t) - \mu^*, g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mu(t)] - g(\mu^*)\rangle \\
&\leq -2\lambda\langle\rho(\mu(t) - \mu^*), g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mu(t)] - \Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mu^*(t)]\rangle, \\
&\leq -2\lambda\sigma\|\mu(t) - \mu^*\|^2 + \lambda\|g(\mu(t)) - g(\mu^*)\|^2 \\
&\quad +\lambda\|\Pi_{\Omega(\mu)}[\mu(t) - \rho\mu(t)] - \Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mu^*(t)]\|^2
\end{aligned} \tag{6.6}$$

Using the Lipschitz continuity of the operator g, we have

$$\begin{aligned}
\|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - \Pi_{\Omega(\mu)}[g(\mu^*) - \rho\mu^*]\| &\leq \delta\|g(\mu) - g(\mu^*) - \rho(\mu - \mu^*)\| \\
&\leq \delta(\zeta + \rho)\|\mu - \mu^*\|.
\end{aligned} \tag{6.7}$$

From (6.6) and (6.7), we have

$$\frac{d}{dt}\|\mu(t) - \mu^*\| \leq 2\xi\lambda\|\mu(t) - \mu^*\|,$$

where

$$\xi = (\delta(1 + \rho) - 2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\|e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (6.3) converges globally exponentially to the unique solution of the inverse quasi variational inequality (2.7). □

We use the dynamical system (6.3) to suggest some iterative for solving the inverse quasi variational inequalities (2.7). These methods can be viewed in the sense of Korpelevich [33] and Noor [51, 54] involving the double projection.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (6.3) becomes

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mu\big], \quad \mu(t_0) = \alpha. \tag{6.8}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (6.8), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_{n+1}], \tag{6.9}$$

where h is the step size.

Now, we can suggest the following implicit iterative method for solving the inverse quasi variational inequality (2.7).

Algorithm 6.1. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}\left[g(\mu_n) - \rho\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h}\right].$$

This is an implicit method. Algorithm 6.1 is equivalent to the following two-step method.

Algorithm 6.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g(\mu_n) - \rho\omega_n - \frac{\omega_n - \mu_n}{h}]. \end{aligned}$$

Discretizing (6.8), we now suggest an other implicit iterative method for solving the inverse quasi variational inequality (2.7).

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mu_{n+1}], \tag{6.10}$$

where h is the step size.

This formulation enables us to suggest the two-step iterative method.

Algorithm 6.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\omega(\mu_n)}[g(\mu_n) - \rho\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\left[g(\omega_n) - \rho\omega_n - \frac{\omega_n - \mu_n}{h}\right]. \end{aligned}$$

Discretizing (6.8), we have

$$\frac{\mu_{n+1}-\mu_n}{h} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mu_{n+1}], \tag{6.11}$$

where h is the step size.

This helps us to suggest the following implicit iterative method for solving the problem (2.7).

Algorithm 6.4. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\Big[g(\omega_n) - \rho\omega_n\Big]. \end{aligned}$$

Discretizing (6.8), we propose another implicit iterative method.

$$\frac{\mu_{n+1}-\mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho\mu_{n+1}],$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (2.7).

Algorithm 6.5. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho\mu_{n+1}].$$

From (6.8), we have

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega((1-\alpha)\mu+\alpha\mu)}[g((1-\alpha)\mu + \alpha\mu) - \rho((1-\alpha)\mu + \alpha\mu)], \tag{6.12}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (6.12) and taking $h = 1$, we have

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega((1-\alpha)\mu_n+\alpha\mu_{n-1})}\big[g\big((1-\alpha)\mu_n + \alpha\mu_{n-1}\big) - \rho((1-\alpha)\mu_n + \alpha\mu_{n-1})\big],$$

which is an inertial type iterative method for solving the inverse quasi variational inequality (2.7). Using the predictor-corrector techniques, we have

Algorithm 6.6. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes*

$$\begin{aligned} \omega_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\omega_n}\big[g(\omega_n) - \rho\omega_n\big], \end{aligned}$$

which is known as the inertial two-step iterative method.

We now introduce the second order dynamical system associated with the inverse quasi variational inequality (2.7). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mu] - g(\mu)\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{6.13}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (6.13) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (6.13) is defined as follows.

Definition 6.4. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (6.13), if,*

$$\gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the inverse quasi variational inequality (2.7), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

From (6.13), we have

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu].$$

Thus, we can rewrite (6.13) as follows:

$$g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mu + \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\big]. \tag{6.14}$$

For $\lambda = 1$, the problem (6.13) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mu\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{6.15}$$

The problem (6.15) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the inverse quasi variational inequalities. Consequently, one can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the inverse quasi variational inequalities and related optimization problems.

We discretize the second-order dynamical systems (6.15) using central finite difference and backward difference schemes to have

$$\gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_n) = P_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_{n+1}], \tag{6.16}$$

where h is the step size.

If $\gamma = 1, h = 1$, then, from equation(6.16) we have

Algorithm 6.7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n + g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mu_{n+1}],$$

which is the extragradient method for solving the inverse quasi variational inequalities (2.7).

Algorithm 6.7 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 6.8. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho y_n], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant.

In a similar way, we have the following two-step method.

Algorithm 6.9. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho y_n], \end{aligned}$$

which is also called the double projection method for solving the inverse quasi variational inequalities (2.7).

We discretize the second-order dynamical systems (6.3) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_{n+1}) = \Pi_{\Omega(\mu_{n+1})}\big[g(\mu_n) - \rho\mu_{n+1}\big],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the inverse quasi variational inequalities (2.7).

Algorithm 6.10. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_n)}\big[g(\mu_{n+1}) - \rho\mu_{n+1} - \gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h}\big].$$

Algorithm 6.10 is called the hybrid inertial proximal method for solving the inverse quasi variational inequalities and related optimization problems. This is a new proposed method.

Note that, for $\gamma = 0$, Algorithm 6.10 reduces to the following iterative method.

Algorithm 6.11. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_{n+1})}\big[g(\mu_{n+1}) - \rho\mu_{n+1} + \frac{\mu_n - \mu_{n-1}}{h}\big],$$

which is called the inertial double projection method.

We now consider the third order dynamical systems associated with the general quasi variational inequalities of the type (2.7). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu], \quad u(a) = \alpha, \dot{\mu}(a) = \beta, \dot{\mu}(b) = 0 \tag{6.17}$$

where $\gamma > 0, \zeta, \xi$ and $\rho > 0$ are constants. Problem (6.17) is called third order dynamical system associated with inverse quasi variational inequalities (2.7).

The equilibrium point of the dynamical system (6.17) is defined as follows.

Definition 6.5. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (6.13), if,*

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general quasi variational inequality (2.5), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Consequently, the problem (6.3) can be equivalent written as

$$g(\mu) = \Pi_{\Omega(\mu)}\left[g(\mu) - \rho\mu + \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt}\right]. \tag{6.18}$$

We discretize the third-order dynamical systems (6.17) using central finite difference and backward difference schemes to have

$$\begin{aligned}&\gamma\frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3} + \zeta\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}\\ &+\xi\frac{3\mu_n - 4\mu_{n-1} + \mu_{n-2}}{2h} + g(\mu_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho(\mu_{n+1})],\end{aligned} \tag{6.19}$$

where h is the step size.

If $\gamma = 1, h = 1, \zeta = 1, \xi = 1$, then, from equation (6.19) after adjustment, we have

Algorithm 6.12. *For a given μ_0, μ_1, compute u_{n+1} by the iterative scheme*

$$u_{n+1} \quad = \quad \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}\big[g(\mu_n) - \rho\mu_{n+1} + \frac{\mu_{n-1} - 3u_n}{2}\big], \quad n = 0, 1, 2, \ldots$$

which is an inertial type hybrid iterative methods for solving the inverse quasi variational inequalities (2.7).

Remark 6.1. *For appropriate and suitable choice of the operator g, convex-valued set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving inverse quasi variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al. [63, 64], one can discuss the convergence analysis of the proposed methods.*

7 Sensitivity Analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences. The behaviour of such problems as a result of changes in the problem data is always of concern. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving the problems due to these and other reasons. In this section, we study the sensitivity analysis of the inverse quasi variational inequalities, that is, examining how solutions of such problems change when the data of the problems are changed.

We now consider the parametric versions of the problem (2.7). To formulate the problem, let M be an open subset of $\mathcal{H}$ in which the parameter λ takes values. Let $g(\mu, \lambda)$ be given identity operator defined on $\mathcal{H} \times \mathcal{H} \times M$ and take value in $\mathcal{H} \times \mathcal{H}$. From now onward, we denote $g_\lambda(.) \equiv g(., \lambda)$ unless otherwise specified.

The parametric inverse variational inequality problem is to find $(\mu, \lambda) \in \mathcal{H} \times M$ such that

$$\langle \rho\mu + g_\lambda(\mu) - g_\lambda(\mu), \nu - g_\lambda(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu). \tag{7.1}$$

We also assume that, for some $\overline{\lambda} \in M$, problem (7.1) has a unique solution $\overline{\mu}$. From Lemma 3.1, we see that the parametric inverse quasi variational inequalities are equivalent to the fixed point problem:

$$g_\lambda(\mu) = \Pi_{\Omega(\mu)}[g_\lambda(\mu) - \rho\mu],$$

or equivalently

$$\mu = \mu - g_\lambda(\mu) + \Pi_{\Omega(\mu)}[g_\lambda(\mu) - \rho\mu].$$

We now define the mapping F_λ associated with the problem (7.1) as

$$F_\lambda(\mu) = \mu - g_\lambda(\mu) + \Pi_{\Omega(\mu)}[g_\lambda(\mu) - \rho\mu], \quad \forall(\mu,\lambda) \in X \times M. \tag{7.2}$$

We use this equivalence to study the sensitivity analysis of the inverse quasi variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (7.1) has a solution $\overline{\mu}$ and X is a closure of a ball in $\mathcal{H}$ centered at $\overline{\mu}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (7.1) has a unique solution $z(\lambda)$ near $\overline{u}$ and the function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 7.1. *Let $I_\lambda(.)$ be an operator on $X \times M$. Then, the operator $g_\lambda(.)$ is said to :*

(a) Locally strongly monotone with constant $\sigma > 0$, if

$$\langle g_\lambda(\mu) - g_\lambda(\nu), \mu - \nu\rangle \geq \sigma\|\mu - \nu\|^2, \quad \forall\lambda \in M, \mu, \nu \in X.$$

(b) Locally Lipschitz continuous with constant $\zeta > 0$, if

$$\|g_\lambda(\mu) - g_\lambda(\nu)\| \leq \zeta\|\mu - \nu\|, \quad \forall\lambda \in M, \mu, \nu \in X.$$

We consider the case, when the solutions of the parametric inverse quasi variational inequality (7.1) lie in the interior of X. Following the ideas of Dafermos [17] and Noor [50,54], we consider the map $F_\lambda(\mu)$ as defined by (7.2). We have to show that the map $F_\lambda(\mu)$ has a fixed point, which is a solution of the parametric inverse quasi variational inequality (7.1). First of all, we prove that the map $F_\lambda(\mu)$, defined by (7.2), is a contraction map with respect to μ uniformly in $\lambda \in M$.

Lemma 7.1. *Let $g_\lambda(.)$ be a locally strongly monotone with constants $\sigma > 0$ and locally Lipschitz continuous with constants $\zeta > 0$ respectively. If Assumption 2.1 holds and $\forall \mu_1, \mu_2 \in X, \quad \lambda \in M$, we have*

$$\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \leq \theta\|\mu_1 - \mu_2\|,$$

for

$$\rho < 1 - k, \quad k < 1, \tag{7.3}$$

where

$$\theta = \left\{\sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta + \rho\right\} = \{k + \rho\} \tag{7.4}$$

and

$$k = \sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta. \tag{7.5}$$

Proof. In order to prove the existence of a solution of (7.1), it is enough to show that the mapping $F_\lambda(\mu)$, defined by (7.2), is a contraction mapping.

For $\mu_1 \neq \mu_2 \in \mathcal{H}$, and using Assumption 2.1, we have

$$\begin{aligned} \|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\ &\quad +\|\Pi_{\Omega(\mu_1)}[g_\lambda(\mu_1) - \rho\mu_1] - \Pi_{\Omega(u_2)}[g_\lambda(\mu_2) - \rho\mu_2]\| \\ &\quad +\|\Pi_{\Omega(u_1)}[g_\lambda(\mu_1) - \rho\mu_1] - \Pi_{\Omega(u_2)}[g_\lambda(\mu_1) - \rho\mu_1]\| \\ &\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\ &\quad +\eta\|\mu_1 - \mu_2\| + \|g_\lambda(\mu_)) - g_\lambda(\mu_2) - \rho(\mu_1 - \mu_2)\| \\ &\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| + \eta\|\mu_1 - \mu_2\| \\ &\quad +\|g_\lambda(\mu_1) - g_\lambda(\mu_2)\| + \rho\|\mu_1 - \mu_2)\|. \end{aligned} \tag{7.6}$$

Since the operator g is a locally strongly monotone with constant $\sigma > 0$ and locally Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned} ||\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2)||^2 &\leq ||u_1 - u_2||^2 - 2\langle g_\lambda(\mu_1) - g_\lambda(\mu_2), \mu_1 - \mu_2\rangle + ||g_\lambda(\mu_1) - g_\lambda(\mu_2)||^2 \\ &\leq (1 - 2\sigma + \zeta^2)||\mu_1 - \mu_2||^2. \end{aligned} \tag{7.7}$$

From (7.5), (7.6), (7.7) and using the locally Lipschitz continuity of the operator g_λ, we have

$$\begin{aligned} \|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \left\{\eta + \zeta + \sqrt{(1 - 2\sigma + \zeta^2)} + \rho\right\} \|\mu_1 - \mu_2\| \\ &= \theta\|\mu_1 - \mu_2\|, \end{aligned}$$

where

$$\theta = k + \rho.$$

From (7.3), it follows that $\theta < 1$. Thus it follows that the mapping $F_\lambda(\mu)$, defined by (7.2), is a contraction mapping and consequently it has a fixed point, which belongs to $\Omega(\mu)$ satisfying the inverse quasi variational inequality (7.1), the required result. □

Remark 7.1. *From Lemma 3.1, we see that the map $F_\lambda(\mu)$ defined by (7.2) has a unique fixed point $\mu(\lambda)$, that is, $\mu(\lambda) = F_\lambda(\mu)$. Also, by assumption, the function $\overline{\mu}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric general quasi variational inequality (7.1). Again using Lemma 3.1, we see that $\overline{\mu}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(\mu)$ and it is also a fixed point of $F_{\overline{\lambda}}(\mu)$. Consequently, we conclude that*

$$\mu(\overline{\lambda}) = \overline{\mu} = F_{\overline{\lambda}}(\mu(\overline{\lambda})).$$

Using Lemma 3.1, we can prove the continuity of the solution $\mu(\lambda)$ of the parametric inverse quasi variational inequality (7.1) using the technique of Noor [13,14].

Lemma 7.2. *Assume that the operator $g_\lambda(.)$ is locally Lipschitz continuous with respect to the parameter λ. If the operator $g_\lambda(.)$ is locally Lipschitz continuous and the map $\lambda \to P_{K_\lambda} u$ is continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (7.2) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

We now state and prove the main result of this paper and is the motivation our next result.

Theorem 7.1. *Let $\overline{\mu}$ be the solution of the parametric inverse quasi variational inequality (7.1) for $\lambda = \overline{\lambda}$. Let $g_\lambda(\mu)$ be the locally strongly monotone Lipschitz continuous operator for all $\mu, \nu \in X$. If the map $\lambda \to \Pi_{\Omega_\mu}$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric inverse quasi variational inequality (7.2) has a unique solution $\mu(\lambda)$ in the interior of $X, u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemma 7.1, Lemma 7.2 and Remark 7.1. □

8 Generalizations and Applications

In this section, we show that the quasi variational inequalities are equivalent to the strongly nonlinear inverse variational inequalities, see Noor [36].

In many applications, the convex-valued set $\Omega(\mu)$ is of the form:

$$\Omega(\mu) = m(\mu) + \Omega, \tag{8.1}$$

where Ω is a convex set and m is a point-to-point mapping.

Let $\mu \in \Omega(\mu)$ be a solution of the problem (2.5). Then from Lemma 3.1, it follows that $\mu \in \Omega(\mu)$ such that

$$g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mu\big]. \tag{8.2}$$

Combining (8.1) and (8.2), we obtain

$$\begin{aligned} g(\mu) &= \Pi_{\Omega(\eta(\mu)+\Omega)}\big[g(\mu) - \rho\mu\big] \\ &= m(\mu) + \Pi_\Omega\big[g(\mu) - m(\mu) - \rho\mu\big]. \end{aligned}$$

This implies that

$$G(\mu) = \Pi_\Omega\big[G(\mu) - \rho\mu\big],$$

with $G(\mu) = g(\mu) - m(\mu)$, which is equivalent to finding $\mu \in g(\mu) \in \Omega$ such that

$$\langle \rho\mu, G(\nu) - G(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega. \tag{8.3}$$

The inequality of the type (8.3) is called the inverse general variational inequality, which is special case of the general variational inequalities, introduced and investigated by Noor [46] in 1988. It have been shown that odd-order and nonsymmetric obstacle boundary value problems can be studied in the general variational inequalities. For more details, see [46, 54, 73, 74]. Thus all the results proved for inverse quasi variational inequalities continue to hold for inverse general variational inequalities (8.3) with suitable modifications and adjustment. Despite the research activates, very few results are available.

We would like to mention that some of the results obtained and presented in this paper can be extended for more multivalued general quasi variational inequalities. To be more precise, let $C(\mathcal{H})$ be a family of nonempty compact subsets of $\mathcal{H}$. Let $\mathcal{T}, V : \mathcal{H} \longrightarrow C(\mathcal{H})$ be the multivalued operators. For a given nonlinear bifunction $N(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ and operators $g, h : \mathcal{H} \longrightarrow \mathcal{H}$, consider the problem of finding $\mu \in \Omega(\mu), w \in \mathcal{T}(\mu), y \in V(\mu)$ such that

$$\langle \rho N(w, y), h(\nu) - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{8.4}$$

where ρ is a constant, is called the multivalued general quasi variational inequality.

For $N(w, y) = (w, y)$, the problem (8.4) reduces to finding $\mu \in \Omega(\mu), w \in \mathcal{T}(\mu), y \in V(\mu)$ such that

$$\langle \rho(w, y), h(\nu) - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{8.5}$$

is called the multivalued inverse quasi variational inequality. We would like to mention that one can obtain various classes of inverse quasi variational inequalities for appropriate and suitable choices of the operators g, h, and convex-valued set $\Omega(\mu)$.

Note that, if $N(w, y) = \mu, \quad h = I$, then the problem (8.5) is equivalent to find $\mu \in \Omega(\mu)$, such that

$$\langle \rho\mu, \nu - g(\mu) \geq 0 \quad \forall \nu \in \Omega(\mu),$$

which is exactly the inverse quasi variational inequality (2.7).

Using Lemma 3.1, one can prove that the problem (8.5) is equivalent to finding $\mu \in \Omega(\mu)$ such that

$$g(u) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(w, y)] \tag{8.6}$$

which can be written as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(w, y)].$$

Thus one can consider the mapping Φ associated with the problem (8.5) as

$$\Phi(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(w, y)],$$

which can be used to discuss the uniqueness of the solution and to propose iterative methods for the problem (8.5).

From (8.4) and (8.6), it follows that the multivalued inverse quasi variational inequalities are equivalent to the fixed problems. Consequently, all results obtained for the problem (2.7) continue to hold for the problem (8.5) with suitable modifications and adjustments. Applying the technique and idea of this paper, similar results can be established for solving system of inverse quasi variational inequalities considered with appropriate modifications. The development of efficient implementable numerical methods for solving the multivalued general quasi variational inequalities and non optimization problems requires further efforts.

Conclusion

In this paper, we have used the equivalence between the inverse quasi variational inequalities and fixed point problems to suggest some new multi step multi-step iterative methods for solving the quasi variational inequalities. These new methods include extragradient methods, modified double projection methods and inertial type are suggested using the techniques of projection method, Wiener-Hopf equations, auxiliary techniques and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. It is an open problem to compare these proposed methods with other methods. Sensitivity analysis is also investigated for inverse quasi variational inequalities using the equivalent fixed point approach. Applying the technique and ideas of Ashish et al. [4, 5], Cho et al. [12] and Kwuni et al. [34], can one explore the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. This is an open problem, which deserves further research efforts. We have shown that the inverse quasi variational inequalities are equivalent to the strongly general variational inequalities under suitable conditions of the convex-valued set. Applications of the fuzzy set theory [52], stochastic [6], quantum calculus, fractal, logistic map [90], fractional and random traffic equilibrium [6] can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research, green energy [39] and variational inequalities. One may explore these aspects of the inverse quasi variational inequality and its variant forms.

Data availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study

Conflict interest

Authors have no conflict of interest.

Authors contributions

All authors contributed equally to the conception, design of the work, analysis, interpretation of data, reviewing it critically and final approval of the version for publication.

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References

[1] Alvarez, F. (2003). Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. *SIAM Journal on Optimization, 14*(2), 773–782. https://doi.org/10.1137/S1052623403427859

[2] Alshejary, A. A., Noor, M. A., & Noor, K. I. (2024). Recent developments in general quasi variational. *International Journal of Analysis and Applications, 22*. https://doi.org/10.28924/2291-8639-22-2024-84

[3] Ames, W. F. (1992). *Numerical Methods for Partial Differential Equations*. Academic Press.

[4] Ashish, K., Rani, M., & Chugh, R. (2014). Julia sets and Mandelbrot sets in Noor orbit. *Applied Mathematics and Computation, 228*(1), 615–631. https://doi.org/10.1016/j.amc.2013.11.077

[5] Ashish, K., Chugh, R., & Rani, M. (2021). *Fractals and Chaos in Noor Orbit: A Four-Step Feedback Approach*. LAP LAMBERT Academic Publishing.

[6] Ashish, K., Cao, J., & Noor, M. A. (2023). Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia. *Journal of Applied Analysis and Computation, 13*(5), 2452–2470. https://doi.org/10.11948/20220350

[7] Aussel, D., Gupta, R., & Mehrab, A. (2013). Gap functions and error bounds for inverse quasi-variational inequality problems. *Journal of Mathematical Analysis and Applications, 407*(2), 270–280. https://doi.org/10.1016/j.jmaa.2013.03.049

[8] Barbagallo, A., & Lo Bainco, S. G. (2024). A random elastic traffic equilibrium problem via stochastic quasi-variational inequalities. *Communications in Nonlinear Science and Numerical Simulation, 131*, 107798. https://doi.org/10.1016/j.cnsns.2023.107798

[9] Bensoussan, A., & Lions, J. L. (1978). *Application des inégalités variationnelles en contrôle stochastique*. Bordas (Dunod).

[10] Bnouhachem, A., Noor, M. A., Khalfaoui, M., & Zhaohan, S. (2011). A self-adaptive projection method for a class of variant variational inequalities. *Journal of Mathematical Inequalities, 5*(1), 117–129.

[11] Chairatsiripong, C., & Thianwan, T. (2022). Novel Noor iterations technique for solving nonlinear equations. *AIMS Mathematics, 7*(6), 10958–10976. https://doi.org/10.3934/math.2022612

[12] Cho, S. Y., Shahid, A. A., Nazeer, W., & Kang, S. M. (2016). Fixed point results for fractal generation in Noor orbit and s-convexity. *SpringerPlus, 5*, 1843. https://doi.org/10.1186/s40064-016-3530-5

[13] Cottle, R. W., Pang, J.-S., & Stone, R. E. (2009). *The Linear Complementarity Problem*. SIAM.

[14] Cristescu, G., & Lupsa, L. (2002). *Non-Connected Convexities and Applications*. Kluwer Academic Publishers. https://doi.org/10.1007/978-1-4615-0003-2

[15] Cristescu, G., & Gianu, G. (2009). Detecting the non-convex sets with Youness and Noor types convexities. *Buletin Stiintific al Universitatii Politehnica Timisoara, Matematica-Fizica, 54*(68)(2).

[16] Cristescu, G., & Mihail, G. (2009). Shape properties of Noor's g-convex sets. In *Proceedings of the Twelfth Symposium of Mathematical Applications*, Timisoara, Romania, 1–13.

[17] Dafermos, S. (1988). Sensitivity analysis in variational inequalities. *Mathematics of Operations Research, 13*, 421–434. https://doi.org/10.1287/moor.13.3.421

[18] Dey, S., & Reich, S. (2023). A dynamical system for solving inverse quasi-variational inequalities. *Optimization, 72*. https://doi.org/10.1080/02331934.2023.21735

[19] Dupuis, P., & Nagurney, A. (1993). Dynamical systems and variational inequalities. *Annals of Operations Research, 44*, 7–42. https://doi.org/10.1007/BF02073589

[20] Glowinski, R., Lions, J. L., & Tremolieres, R. (1981). *Numerical Analysis of Variational Inequalities*. North-Holland.

[21] Glowinski, R., & Le Tallec, P. (1989). *Augmented Lagrangian and Operator Splitting Methods in Nonlinear Mechanics*. SIAM. https://doi.org/10.1137/1.9781611970838

[22] Han, Y., Huang, N., Lu, J., & Xiao, Y. (2017). Existence and stability of solutions to inverse variational inequality problems. *Applied Mathematics and Mechanics (English Edition), 38*(5), 749–764. https://doi.org/10.1007/s10483-017-2191-9

[23] He, B. S. (1999). A Goldstein's type projection method for a class of variant variational inequalities. *Journal of Computational Mathematics, 17*(4), 425–434.

[24] He, B., He, X., & Liu, H. X. (2010). Solving a class of constrained 'black-box' inverse variational inequalities. *European Journal of Operational Research, 204*, 391–401. https://doi.org/10.1016/j.ejor.2009.07.006

[25] He, X., & Liu, H. X. (2011). Inverse variational inequalities with projection-based solution methods. *European Journal of Operational Research, 208*, 12–18. https://doi.org/10.1016/j.ejor.2010.08.022

[26] He, S., & Dong, Q. L. (2018). An existence-uniqueness theorem and alternating contraction projection methods for inverse variational inequalities. *Journal of Inequalities and Applications, 2018*, Article 351. https://doi.org/10.1186/s13660-018-1943-0

[27] He, B. S., Liu, H. X., Li, M., & He, X. Z. (2006). PPA-based methods for monotone inverse variational inequalities. *Sciencepaper Online*. http://www.paper.edu.cn

[28] Jabeen, S., Mohsin, B. B., Noor, M. A., & Noor, K. I. (2021). Inertial projection methods for solving general quasi-variational inequalities. *AIMS Mathematics, 6*(2), 1075–1086. https://doi.org/10.3934/math.2021064

[29] Jabeen, S., Noor, M. A., & Noor, K. I. (2022). Some inertial methods for solving a system of quasi-variational inequalities. *Journal of Advanced Mathematical Studies, 15*(1), 1–10.

[30] Jabeen, S., Macías-Díaz, J. E., Noor, M. A., Khan, M. B., & Noor, K. I. (2022). Design and convergence analysis of some implicit inertial methods for quasi-variational inequalities via the Wiener-Hopf equations. *Applied Numerical Mathematics, 182*, 76–86. https://doi.org/10.1016/j.apnum.2022.08.001

[31] Khan, A. G., Noor, M. A., & Noor, K. I. (2015). Dynamical systems for general quasi-variational inequalities. *Annals of Functional Analysis, 6*(1), 193–209. https://doi.org/10.15352/afa/06-1-14

[32] Kinderlehrer, D., & Stampacchia, G. (2000). *An Introduction to Variational Inequalities and Their Applications*. SIAM.

[33] Korpelevich, G. M. (1976). The extragradient method for finding saddle points and other problems. *Ekonomika i Matematicheskie Metody, 12*, 747–756.

[34] Kwuni, Y. C., Shahid, A. A., Nazeer, W., Butt, S. I., Abbas, M., & Kang, S. M. (2019). Tricorns and multicorns in Noor Orbit with s-convexity. *IEEE Access, 7*. https://doi.org/10.1109/ACCESS.2019.2928796

[35] Lions, J., & Stampacchia, G. (1967). Variational inequalities. *Communications on Pure and Applied Mathematics, 20*, 493–519. https://doi.org/10.1002/cpa.3160200302

[36] Lemke, C. F. (1965). Bimatrix equilibrium points and mathematical programming. *Management Science, 11*, 681–689. https://doi.org/10.1287/mnsc.11.7.681

[37] Mijajlovic, N., Milojica, J., & Noor, M. A. (2019). Gradient-type projection methods for quasi-variational inequalities. *Optimization Letters, 13*, 1885–1896. https://doi.org/10.1007/s11590-018-1323-1

[38] Nagurney, A., & Zhang, D. (1996). *Projected Dynamical Systems and Variational Inequalities with Applications*. Kluwer Academic Publishers.

[39] Natarajan, S. K., & Negi, D. (2024). Green innovations utilizing fractal and power for solar panel optimization. In R. Sharma, G. Rana, & S. Agarwal (Eds.), *Green Innovations for Industrial Development and Business Sustainability* (pp. 146–152). CRC Press. https://doi.org/10.1201/9781003458944-10

[40] Niculescu, C. P., & Persson, L. E. (2018). *Convex Functions and Their Applications*. Springer-Verlag. https://doi.org/10.1007/978-3-319-78337-6_1

[41] Noor, M. A. (1971). *Riesz-Fréchet Theorem and Monotonicity* (Master's thesis). Queen's University, Kingston, Ontario, Canada.

[42] Noor, M. A. (1975). *On Variational Inequalities* (Doctoral dissertation). Brunel University, London, U.K.

[43] Noor, M. A. (1985). An iterative scheme for a class of quasi-variational inequalities. *Journal of Mathematical Analysis and Applications, 110*, 463–468. https://doi.org/10.1016/0022-247X(85)90308-7

[44] Noor, M. A. (1986). Generalized quasi-complementarity problems. *Journal of Mathematical Analysis and Applications, 120*, 321–327. https://doi.org/10.1016/0022-247X(86)90219-2

[45] Noor, M. A. (1988). The quasi-complementarity problem. *Journal of Mathematical Analysis and Applications, 130*, 344–353. https://doi.org/10.1016/0022-247X(88)90310-1

[46] Noor, M. A. (1988). General variational inequalities. *Applied Mathematics Letters, 1*(2), 119–121. https://doi.org/10.1016/0893-9659(88)90054-7

[47] Noor, M. A. (1988). Quasi variational inequalities. *Applied Mathematics Letters, 1*(4), 367–370. https://doi.org/10.1016/0893-9659(88)90152-8

[48] Noor, M. A. (1988). Fixed point approach for complementarity problems. *Journal of Mathematical Analysis and Applications, 133*, 437–448. https://doi.org/10.1016/0022-247X(88)90413-1

[49] Noor, M. A. (1992). General algorithm for variational inequalities. *Journal of Optimization Theory and Applications, 73*, 409–413. https://doi.org/10.1007/BF00940189

[50] Noor, M. A. (1997). Sensitivity analysis for quasi variational inequalities. *Journal of Optimization Theory and Applications, 95*(2), 399–407. https://doi.org/10.1023/A:1022691322968

[51] Noor, M. A. (2000). New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications, 251*, 217–230. https://doi.org/10.1006/jmaa.2000.7042

[52] Noor, M. A. (2000). Variational inequalities for fuzzy mappings (III). *Fuzzy Sets and Systems, 110*(1), 101–108. https://doi.org/10.1016/S0165-0114(98)00131-6

[53] Noor, M. A. (2002). A Wiener-Hopf dynamical system and variational inequalities. *New Zealand Journal of Mathematics, 31*, 173–182.

[54] Noor, M. A. (2004). Some developments in general variational inequalities. *Applied Mathematics and Computation, 152*, 199–277. https://doi.org/10.1016/S0096-3003(03)00558-7

[55] Noor, M. A. (2002). Stability of the modified projected dynamical systems. *Computers & Mathematics with Applications, 44*, 1–5. https://doi.org/10.1016/S0898-1221(02)00125-6

[56] Noor, M. A. (2007). Merit functions for quasi variational inequalities. *Journal of Mathematical Inequalities, 1*, 259–269.

[57] Noor, M. A. (2008). Differentiable nonconvex functions and general variational inequalities. *Applied Mathematics and Computation, 199*(1-2), 623–630. https://doi.org/10.1016/j.amc.2007.10.023

[58] Noor, M. A. (2009). Extended general variational inequalities. *Applied Mathematics Letters, 22*, 182–185. https://doi.org/10.1016/j.aml.2008.03.007

[59] Noor, M. A. (2009). *Principles of Variational Inequalities*. LAP Lambert Academic Publishing AG & Co., Saarbrücken, Germany.

[60] Noor, M. A. (2012). On general quasi variational inequalities. *Journal of King Saud University - Science, 24*, 81–88. https://doi.org/10.1016/j.jksus.2010.07.002

[61] Noor, M. A., & Al-Said, E. (1999). Change of variable method for generalized complementarity problems. *Journal of Optimization Theory and Applications, 100*, 389–395. https://doi.org/10.1023/A:1021790404792

[62] Noor, M. A., & Noor, K. I. (1999). Sensitivity analysis for quasi-variational inclusions. *Journal of Mathematical Analysis and Applications, 236*, 290–299. https://doi.org/10.1006/jmaa.1999.6424

[63] Noor, M. A., & Noor, K. I. (2022). Some novel aspects of quasi variational inequalities. *Earthline Journal of Mathematical Sciences, 10*(1), 1–66. https://doi.org/10.34198/ejms.10122.166

[64] Noor, M. A., & Noor, K. I. (2022). Dynamical system technique for solving quasi variational inequalities. *U.P.B. Scientific Bulletin, Series A, 84*(4), 55–66.

[65] Noor, M. A., & Noor, K. I. (2024). Some new iterative schemes for solving general quasi variational inequalities. *Le Matematiche, 79*(2), 327–370.

[66] Noor, M. A., & Noor, K. I. (2024). Some novel aspects and applications of Noor iterations and Noor orbits. *Journal of Advanced Mathematical Studies, 17*(3), 276–284.

[67] Noor, M. A., & Noor, K. I. (2025). New iterative methods for inverse variational inequalities. *Journal of Advanced Mathematical Studies, 18*, in press.

[68] Noor, M. A., Noor, K. I., & Khan, A. G. (2013). Some iterative schemes for solving extended general quasi variational inequalities. *Applied Mathematics & Information Sciences, 7*(3), 917–925. https://doi.org/10.12785/amis/070309

[69] Noor, M. A., Noor, K. I., & Khan, A. G. (2015). Dynamical systems for quasi variational inequalities. *Annals of Functional Analysis, 6*(1), 193–209. https://doi.org/10.15352/afa/06-1-14

[70] Noor, M. A., Noor, K. I., Khan, A. G., & Ghiura, A. (2010). Iterative algorithms for solving a class of quasi variational inequalities. *U.P.B. Scientific Bulletin, Series A, 72*(1), 2–13.

[71] Noor, M. A., Noor, K. I., & Mohsen, B. B. (2021). Some new classes of general quasi variational inequalities. *AIMS Mathematics, 6*(6), 6406–6421. https://doi.org/10.3934/math.2021376

[72] Noor, M. A., Noor, K. I., & Bnouhachem, A. (2021). Some new iterative methods for variational inequalities. *Canadian Journal of Applied Mathematics, 3*(1), 1–17.

[73] Noor, M. A., Noor, K. I., & Rassias, M. T. (2020). New trends in general variational inequalities. *Acta Applicandae Mathematicae, 170*(1), 981–1046. https://doi.org/10.1007/s10440-020-00366-2

[74] Noor, M. A., Noor, K. I., & Rassias, Th. M. (1993). Some aspects of variational inequalities. *Journal of Computational and Applied Mathematics, 47*, 285–312. https://doi.org/10.1016/0377-0427(93)90058-J

[75] Noor, M. A., Noor, K. I., & Rassias, Th. M. (2010). Parametric general quasi variational inequalities. *Mathematical Communications, 15*(1), 205–212.

[76] Noor, M. A., & Oettli, W. (1994). On general nonlinear complementarity problems and quasi equilibria. *Le Matematiche, 49*, 313–331.

[77] Patriksson, M. (1998). *Nonlinear Programming and Variational Inequalities: A Unified Approach*. Kluwer Academic Publishers. https://doi.org/10.1007/978-1-4757-2991-7

[78] Robinson, S. M. (1992). Normal maps induced by linear transformations. *Mathematics of Operations Research, 17*, 691–714. https://doi.org/10.1287/moor.17.3.691

[79] Sanaullah, K., Ullah, S., & Aloraini, N. M. (2024). A self-adaptive three-step numerical scheme for variational inequalities. *Axioms, 13*, 57. https://doi.org/10.3390/axioms13010057

[80] Shi, P. (1991). Equivalence of variational inequalities with Wiener-Hopf equations. *Proceedings of the American Mathematical Society, 111*, 339–346. https://doi.org/10.1090/S0002-9939-1991-1037224-3

[81] Shehu, Y., Gibali, A., & Sagratella, S. (2020). Inertial projection-type method for solving quasi variational inequalities in real Hilbert space. *Journal of Optimization Theory and Applications, 184*, 877–894. https://doi.org/10.1007/s10957-019-01616-6

[82] Stampacchia, G. (1964). Formes bilineaires coercitives sur les ensembles convexes. *Comptes Rendus de l'Académie des Sciences de Paris, 258*, 4413–4416.

[83] Suantai, S., Noor, M. A., Kankam, K., & Cholamjiak, P. (2021). Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting. *Advances in Difference Equations, 2021*, Article ID 265. https://doi.org/10.1186/s13662-021-03422-9

[84] Trinh, T. Q., & Vuong, P. T. (2024). The projection algorithm for inverse quasi-variational inequalities with applications to traffic assignment and network equilibrium control problems. *Optimization*, 2024, 1–25. https://doi.org/10.1080/02331934.2024.2329788

[85] Tseng, P. (2000). A modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization, 38*, 431–446. https://doi.org/10.1137/S0363012998338806

[86] Vuong, P. T., He, X., & Thong, D. V. (2021). Global exponential stability of a neural network for inverse variational inequalities. *Journal of Optimization Theory and Applications, 190*, 915–930. https://doi.org/10.1007/s10957-021-01915-x

[87] Xia, Y. S., & Wang, J. (2000). A recurrent neural network for solving linear projection equations. *Neural Networks, 13*, 337–350. https://doi.org/10.1016/S0893-6080(00)00019-8

[88] Xia, Y. S., & Wang, J. (2000). On the stability of globally projected dynamical systems. *Journal of Optimization Theory and Applications, 106*, 129–150. https://doi.org/10.1023/A:1004611224835

[89] Youness, E. A. (1999). E-convex sets, E-convex functions and E-convex programming. *Journal of Optimization Theory and Applications, 102*(2), 439–450. https://doi.org/10.1023/A:1021792726715

[90] Yadav, A., & Jha, K. (2016). Parrondo's paradox in the Noor logistic map. *International Journal of Advanced Research in Engineering and Technology, 7*(5), 01–06.