

On the Solution of Fractional Option Pricing Model by Convolution Theorem

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Abstract

The classical Black-Scholes equation driven by Brownian motion has no memory, therefore it is proper to replace the Brownian motion with fractional Brownian motion (FBM) which has long-memory due to the presence of the Hurst exponent. In this paper, the option pricing equation modeled by fractional Brownian motion is obtained. It is further reduced to a one-dimensional heat equation using Fourier transform and then a solution is obtained by applying the convolution theorem.

1. Introduction

In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or after effect, that is, there is the effect of delay on state equations. In many mathematical models, the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems depend not only on present and past states but also contain the derivatives with delays. In such cases, class of stochastic differential equations driven by fractional Brownian motion provides an important tool for describing and analyzing

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such systems. Fractional Brownian motion of Hurst exponent $H \in (0, 1)$ is a stochastic process $\{B_H(t), t \in \mathbb{R}\}$ which satisfies the following;

- I. $B_H(t)$ is Gaussian, that is, for every $t > 0$, $B_H(t)$ has a normal distribution.
- II. $B_H(t)$ is a self similar process meaning that for any $\alpha > 0$, $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$.
- III. It has stationary increments, that is, $B_H(t) - B_H(s) \sim B_H(t - s)$.

Fractional Brownian motion was first introduced by Kolmogorov in 1940 and then Mandelbrot and Van Ness discovered many of its properties.

Fractional Brownian motion can be applied in pricing financial derivatives. A financial derivative is an instrument whose price depends on, or is derived from the value of another asset. Often, this underlying asset is a stock. The concept of financial derivative is not new. While there remains some historical debate as to the exact date of the creation of financial derivatives, it is well accepted that the first attempt at modern derivative pricing began with the work of Charles Castelly, published in 1877. In 1969, Fisher Black and Myron Scholes got an idea that would change the world of finance forever. The central idea of their paper revolved around the discovery that one did not need to estimate the expected return of a stock in order to price an option written on that stock.

The Black-Scholes option pricing equation modeled by fractional Brownian motion is derived by replacing the standard Brownian motion involved in the classical Black-Scholes model with fractional Brownian motion which contains the Hurst exponent H . The Hurst exponent denoted by H , is a statistical measure used to classify time series. The value of H varies between 0 and 1.

In this paper we intend to reduce the Black-Scholes option pricing equation modeled by fractional Brownian motion to a one-dimensional heat equation using Fourier transform and then obtain the solution by applying the convolution theorem.

2. Fractional Option Pricing Model

Theorem 2.1. *Let a generic payoff function $G(t) = V(S, t)$. Then the partial differential equation associated with the price of the derivative on the stock price is*

$$\frac{\partial V}{\partial t} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad S > 0, \quad t > 0, \quad (2.1)$$

$$H \in (0, 1), \quad H \neq \frac{1}{2},$$

where V is the call option price, t is the time to maturity, H is the Hurst exponent, σ is the volatility, S is the stock price and r is the discount rate.

Proof. The stock price S_t follows the fractional Brownian motion process

$$dS = \mu S dt + \sigma S dB_H(t). \quad (2.2)$$

The wealth of an investor X_t follows a diffusion process given by

$$dX = \Lambda dS + r(X - \Lambda S) dt. \quad (2.3)$$

Putting equation (2.2) into equation (2.3) yields

$$dX = \{rX + \Lambda S(\mu - r)\} dt + \Lambda S \sigma dB_H(t), \quad (2.4)$$

where $\mu - r$ is the risk premium.

Suppose that the value of this claim at time t is given by

$$G(t) = V(S, t), \quad S = S_t. \quad (2.5)$$

Applying the Ito's formula for fractional Brownian motion on equation (2.5), we have

$$dG = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + Ht^{2H-1} \frac{\partial^2 V}{\partial S^2} (dS)^2. \quad (2.6)$$

Substituting (2.2) in (2.6), we have

$$dG = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} [\mu S dt + \sigma S dB_H(t)] + Ht^{2H-1} \frac{\partial^2 V}{\partial S^2} [\mu S dt + \sigma S dB_H(t)]^2 \quad (2.7)$$

$$\begin{aligned} \Rightarrow dG &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} [\mu S dt + \sigma S dB_H(t)] \\ &+ Ht^{2H-1} \frac{\partial^2 V}{\partial S^2} [\mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 dt dB_H(t) + \sigma^2 S^2 (dB_H(t))^2]. \end{aligned} \quad (2.8)$$

Multiplication rule implies:

$$(dt)^2 = 0; dt dB_H(t) = 0; (dB_H(t))^2 = dt.$$

See Bernard [5].

Thus, (2.8) reduces to

$$dG = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} [\mu S dt + \sigma S dB_H(t)] + Ht^{2H-1} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt. \quad (2.9)$$

Collecting like terms, we have

$$dG = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + Ht^{2H-1} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt + \sigma S \frac{\partial V}{\partial S} dB_H(t). \quad (2.10)$$

Using (3.5), we have

$$dV = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + Ht^{2H-1} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt + \sigma S \frac{\partial V}{\partial S} dB_H(t). \quad (2.11)$$

We have under the assumption of complete market that

$$X(t) = V(S, t), \quad \forall t \in (0, T)$$

$$\Rightarrow dX = dV.$$

Thus, equating coefficients, we have

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 V}{\partial S^2} = rV + \Lambda_t S(\mu - r) \quad (2.12)$$

$$\text{and} \quad \sigma S \frac{\partial V}{\partial S} = \Lambda_t \sigma S, \quad (2.13)$$

$$\Lambda_t = \frac{\partial V}{\partial S}. \quad (2.14)$$

Substituting equation (2.14) into equation (2.12), we have

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 V}{\partial S^2} = rV + S\mu \frac{\partial V}{\partial S} - Sr \frac{\partial V}{\partial S}. \quad (2.15)$$

This implies that

$$\frac{\partial V}{\partial t} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 V}{\partial S^2} + Sr \frac{\partial V}{\partial S} - rV = 0.$$

$V(S, t)$ is the European call option price, S is the stock price at time t , t is the time to the expiration of the option, r is the discount rate, σ represents the volatility function of the underlying asset and H is the Hurst exponent.

3. The Model

Theorem 3.1. *Let equation (2.1) be given by*

$$\frac{\partial V}{\partial t} + Ht^{2H-1} S^2 \sigma^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad S > 0, t > 0, \tag{3.1}$$

with $V(0, t) = 0$, $V(S, t) \sim S$ as $S \rightarrow \infty$, $V(S, T) = \max\{|S - K|, 0\}$.

Then (3.1) can be reduced to a one-dimensional heat equation of the form

$$\frac{\partial u}{\partial \tau} = p \frac{\partial^2 u}{\partial x^2}. \tag{3.2}$$

Proof. Set $\tau = \frac{\sigma^2(T-t)}{2}$; $x = \ln(S/K)$ and

$$V(S, t) = Kv(x, \tau). \tag{3.3}$$

Differentiating (3.3), we have

$$\frac{\partial V}{\partial t} = K \frac{\partial v}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \left(K \frac{\partial v}{\partial \tau} \right) \left(-\frac{\sigma^2}{2} \right), \tag{3.4}$$

$$\frac{\partial V}{\partial S} = K \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial S} = \left(K \frac{\partial v}{\partial x} \right) \left(\frac{1}{S} \right) = \frac{K}{S} \frac{\partial v}{\partial x}, \tag{3.5}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S} \left(\frac{\partial V}{\partial S} \right) = \frac{\partial}{\partial S} \left(\frac{K}{S} \frac{\partial v}{\partial x} \right) = \frac{K}{S} \left(\frac{\partial}{\partial S} \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \left(\frac{\partial}{\partial S} \frac{K}{S} \right) \\ &= \frac{K}{S} \left(\frac{\partial}{\partial S} \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \left(-\frac{K}{S^2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{K}{S} \left[\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) \frac{dx}{dS} \right] - \frac{K}{S^2} \frac{\partial v}{\partial x} \\
&= \frac{K}{S} \frac{\partial^2 v}{\partial x^2} \left(\frac{1}{S} \right) - \frac{K}{S^2} \frac{\partial v}{\partial x}, \\
\frac{\partial^2 V}{\partial S^2} &= -\frac{K}{S^2} \frac{\partial v}{\partial x} + \frac{K}{S^2} \frac{\partial^2 v}{\partial x^2}. \tag{3.6a}
\end{aligned}$$

The terminal condition is

$$\begin{aligned}
V(S, T) &= \max\{S - K, 0\} \\
&= \max\{Ke^x - K, 0\}.
\end{aligned}$$

Let

$$\begin{aligned}
V(S, T) &= Kv(x, 0) \\
\Rightarrow v(x, 0) &= \max\{e^x - 1, 0\}. \tag{3.6b}
\end{aligned}$$

Substitute (3.4), (3.5) and (3.6a) in (3.1)

$$\begin{aligned}
&\left(K \frac{\partial v}{\partial \tau} \right) \left(-\frac{\sigma^2}{2} \right) + H \left(T - \frac{2\tau}{\sigma^2} \right)^{2H-1} S^2 \sigma^2 \left(-\frac{K}{S^2} \frac{\partial v}{\partial x} + \frac{K}{S^2} \frac{\partial^2 v}{\partial x^2} \right) \\
&+ rS \left(\frac{K}{S} \frac{\partial v}{\partial x} \right) - rKv = 0. \tag{3.7}
\end{aligned}$$

Let $m = \frac{2\tau}{\sigma^2}$. Then we have

$$-\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau} + H(T - m)^{2H-1} S^2 \sigma^2 \left(\frac{-1}{S^2} \frac{\partial v}{\partial x} + \frac{1}{S^2} \frac{\partial^2 v}{\partial x^2} \right) + rS \left(\frac{1}{S} \frac{\partial v}{\partial x} \right) - rv = 0, \tag{3.8}$$

$$-\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau} - H(T - m)^{2H-1} \sigma^2 \frac{\partial v}{\partial x} + H(T - m)^{2H-1} \sigma^2 \frac{\partial^2 v}{\partial x^2} + r \frac{\partial v}{\partial x} - rv = 0, \tag{3.9}$$

$$\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau} + H(T - m)^{2H-1} \sigma^2 \frac{\partial v}{\partial x} - r \frac{\partial v}{\partial x} - H(T - m)^{2H-1} \sigma^2 \frac{\partial^2 v}{\partial x^2} + rv = 0, \tag{3.10}$$

$$\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau} + [H(T - m)^{2H-1} \sigma^2 - r] \frac{\partial v}{\partial x} - H(T - m)^{2H-1} \sigma^2 \frac{\partial^2 v}{\partial x^2} + rv = 0, \quad (3.11)$$

$$\frac{\partial v}{\partial \tau} + \left[2H(T - m)^{2H-1} - \frac{2r}{\sigma^2} \right] \frac{\partial v}{\partial x} - 2H(T - m)^{2H-1} \frac{\partial^2 v}{\partial x^2} + \frac{2rv}{\sigma^2} = 0, \quad (3.12)$$

Let $p = 2H(T - m)^{2H-1}$ and $q = \frac{2r}{\sigma^2}$. Then we have

$$\frac{\partial v}{\partial \tau} + (p - q) \frac{\partial v}{\partial x} - p \frac{\partial^2 v}{\partial x^2} + qv = 0, \quad (3.13)$$

$$\frac{\partial v}{\partial \tau} = p \frac{\partial^2 v}{\partial x^2} + (q - p) \frac{\partial v}{\partial x} - qv. \quad (3.14a)$$

Let
$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau). \quad (3.14b)$$

Using the product rule, we have

$$v_\tau = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_\tau,$$

$$v_x = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_x,$$

$$v_{xx} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} u_x + e^{\alpha x + \beta \tau} u_{xx},$$

where v_τ and v_x stand for the first partial derivative of v with respect to τ and x respectively. u_τ and u_x stand for the first partial derivative of u with respect to τ and x respectively. v_{xx} and u_{xx} stand for the second partial derivative of v and u with respect to x .

Substituting into (3.14a), we have

$$\begin{aligned} & \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_\tau \\ &= p[\alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} u_x + e^{\alpha x + \beta \tau} u_{xx}] + (q - p)[\alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_x] \\ & \quad - qe^{\alpha x + \beta \tau} u. \end{aligned}$$

Simplifying, we have

$$\begin{aligned}\beta u + u_\tau &= p[\alpha^2 u + 2\alpha u_x + u_{xx}] + (q - p)(\alpha u + u_x) - qu, \\ u_\tau &= pu_{xx} + [2\alpha p + (q - p)]u_x + [\alpha^2 p + (q - p)\alpha - q - \beta]u.\end{aligned}\quad (3.15)$$

Choose $\alpha = \frac{p - q}{2p}$ and $\beta = \frac{-(p + q)^2}{4p}$.

Thus (3.15) is reduced to

$$u_\tau = pu_{xx}.\quad (3.16)$$

Equation (3.16) is a one-dimensional heat equation. We shall solve (3.16) using Fourier transform.

Solution to a One Dimensional Heat Equation

Let

$$\frac{\partial u}{\partial \tau} = p \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\quad (3.17a)$$

$$u(x, 0) = f(x),\quad (3.17b)$$

$$u(x, \tau) \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \tau \rightarrow \pm\infty.\quad (3.17c)$$

Take Fourier transform of both sides of (3.17a).

$$\mathcal{F} \frac{\partial u}{\partial \tau} = p \mathcal{F} \frac{\partial^2 u}{\partial x^2},\quad (3.18)$$

where we have acknowledged the linearity of the Fourier transform in moving the constant p out of the transform. Recall the definition of the Fourier transform of a function $u(x, \tau)$:

$$\mathcal{F}(u) = U(w, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, \tau) e^{iwx} dx,\quad (3.19)$$

see Bracewell [1].

This implies that

$$\mathcal{F}\left(\frac{\partial u}{\partial \tau}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial \tau}(x, \tau) e^{iwx} dx. \tag{3.20}$$

The only τ -dependence in the integral on the right is in the integrand $u(x, \tau)$. As a result, we may write

$$\begin{aligned} \mathcal{F}\left(\frac{\partial u}{\partial \tau}\right) &= \frac{d}{d\tau} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, \tau) e^{iwx} dx \right], \\ \mathcal{F}\left(\frac{\partial u}{\partial \tau}\right) &= \frac{d}{d\tau} U(w, \tau). \end{aligned} \tag{3.21}$$

In other words, the partial time-derivative of $u(x, \tau)$ is simply the total time-derivative of $U(w, \tau)$.

Similarly

$$\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x, \tau) e^{iwx} dx. \tag{3.22}$$

Using integration by parts:

Set $f = e^{iwx}$ and $dg = \frac{\partial u}{\partial x} dx$ to get;

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x, \tau) e^{iwx} dx = \frac{1}{\sqrt{2\pi}} e^{iwx} u(x, \tau) \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} (iw) \int_{-\infty}^{\infty} u(x, \tau) e^{iwx} dx. \tag{3.23}$$

Recall that the boundary conditions for the heat equation on the infinite interval were:

$$u(x, \tau) \rightarrow 0 \text{ as } x \rightarrow \pm\infty.$$

As a result, the contributions to the first term vanish and we are left with the integral. Notice that the integral is simply a multiple of the Fourier transform of u , that is

$$\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = -iw\mathcal{F}(u) = -iwU(w, \tau). \tag{3.24}$$

We may iterate this result to obtain the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) &= \mathcal{F}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right] \\ &= -iw\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = (-iw)^2\mathcal{F}(u), \\ \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) &= -w^2\mathcal{F}(u) = -w^2U(w, \tau).\end{aligned}\tag{3.25}$$

Substituting (3.21) and (3.25) in equation (3.18), we have

$$\frac{dU(w, \tau)}{d\tau} = -pw^2U(w, \tau).\tag{3.26}$$

The above equation may be solved in the same way as we solve a first order linear ordinary differential equation in τ

$$\frac{dU(w, \tau)}{d\tau} + pw^2U(w, \tau) = 0.\tag{3.27}$$

Note that the integrating factor is $I(\tau) = e^{pw^2\tau}$ to give

$$\frac{d}{d\tau}[e^{pw^2\tau}U(w, \tau)] = 0.\tag{3.28}$$

Integrating partially with respect to τ yields

$$e^{pw^2\tau}U(w, \tau) = C(w),\tag{3.29}$$

$$U(w, \tau) = C(w)e^{-pw^2\tau}.\tag{3.30}$$

Notice that at time $\tau = 0$,

$$U(w, 0) = C(w).\tag{3.31}$$

In other words, $C(w)$ is the Fourier transform of the function $u(x, 0)$,

$$\Rightarrow C(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0)e^{iwx} dx,\tag{3.32}$$

But $u(x, 0) = f(x)$ from (3.17b)

$$\Rightarrow C(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{iwx} dx = F(w), \quad (3.33)$$

where $F(w)$ is the Fourier transform of $f(x)$.

Thus, equation (3.30) may be re-written as:

$$U(w, \tau) = F(w)e^{-pw^2\tau}. \quad (3.34)$$

We will now take the inverse Fourier transform of both sides to retrieve $u(x, \tau)$

Let $G(w, \tau) = e^{-pw^2\tau}$. Thus,

$$U(w, \tau) = F(w)G(w, \tau). \quad (3.35)$$

To get back the solution $u(x, \tau)$ we need to invert $U(w, \tau)$. The inversion is done via convolution theorem for Fourier transform.

4. Convolution Theorem for Fourier Transforms

Let $\mathcal{F}(f) = F$ and $\mathcal{F}(g) = G$. Then we have

$$\mathcal{F}^{-1}(FG) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)g(x-s)ds. \quad (4.1)$$

We will apply the above convolution theorem to equation (3.34). This, of course, means that we will find the inverse Fourier transform of $e^{-pw^2\tau}$,

$$\begin{aligned} \mathcal{F}^{-1}(e^{-pw^2\tau}) &= \frac{1}{\sqrt{2\pi}} \int_R e^{iwx} e^{-pw^2\tau} dw \\ &= \frac{1}{\sqrt{2\pi}} \int_R e^{iwx - pw^2\tau} dw. \end{aligned} \quad (4.2)$$

By completing the squares, we require that

$$iwx - pw^2\tau = A(w + B)^2 + C, \quad (4.3a)$$

where A, B, C are to be determined, see Coppex [2]. Expanding (4.3a), we have

$$iwx - pw^2\tau = Aw^2 + 2AwB + AB^2 + C.$$

Equating the coefficients of the successive powers in w gives the following set of equations for A, B, C :

$$A = -p\tau \tag{4.3b}$$

$$2AB = ix, \tag{4.3c}$$

$$AB^2 + C = 0. \tag{4.3d}$$

The solution of this set of equations is: $A = -p\tau$; $B = \frac{-ix}{2p\tau}$; $C = \frac{-x^2}{4p\tau}$.

Substituting these values into (4.38a), we have

$$iwx - pw^2\tau = (-p\tau)\left(w - \frac{ix}{2p\tau}\right)^2 - \frac{x^2}{4p\tau}.$$

Thus, (4.2) becomes:

$$\begin{aligned} \mathcal{F}^{-1}(e^{-pw^2\tau}) &= \frac{1}{\sqrt{2\pi}} \int_R e^{(-p\tau)\left(w - \frac{ix}{2p\tau}\right)^2 - \frac{x^2}{4p\tau}} dw \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4p\tau}} \int_R e^{(-p\tau)\left(w - \frac{ix}{2p\tau}\right)^2} dw. \end{aligned} \tag{4.4}$$

We further change variables: let $X = \sqrt{p\tau}\left(w - \frac{ix}{2p\tau}\right)$,

$$dX = \sqrt{p\tau} dw.$$

Thus, (4.4) becomes:

$$\mathcal{F}^{-1}(e^{-pw^2\tau}) = \frac{1}{\sqrt{2\pi p\tau}} e^{-\frac{x^2}{4p\tau}} \int_R e^{-X^2} dX. \tag{4.5a}$$

We will evaluate $\int_R e^{-X^2} dX$. Note that if $X = (x_1, x_2) \in R^2$, then on one hand, we have

$$\begin{aligned} \int_{R^2} e^{-X^2} dX &= \int_R e^{-x_1^2} dx_1 \int_R e^{-x_2^2} dx_2, \\ \int_{R^2} e^{-X^2} dX &= \left(\int_R e^{-x^2} dx \right)^2. \end{aligned} \tag{4.5b}$$

On the other hand, using polar co-ordinates $X = (x_1, x_2) = r(\cos \theta, \sin \theta)$,

$$dx_1 dx_2 = r dr d\theta, \quad r \in [0, \infty[, \quad \theta \in [0, 2\pi[.$$

Thus, we have

$$\begin{aligned} \int_{R^2} e^{-X^2} dX &= \int_0^\infty r e^{-r^2} dr \int_0^{2\pi} d\theta \\ &= 2\pi \int_0^\infty r e^{-r^2} dr \\ &= 2\pi \left(\frac{1}{2} \right) = \pi \\ \Rightarrow \int_{R^2} e^{-X^2} dX &= \pi. \end{aligned} \tag{4.5c}$$

Combining (4.5b) and (4.5c), we have

$$\begin{aligned} \left(\int_R e^{-x^2} dx \right)^2 &= \pi \\ \Rightarrow \int_R e^{-x^2} dx &= \sqrt{\pi}. \end{aligned}$$

Substituting in (4.5a), we have

$$\mathcal{F}^{-1}(e^{-pw^2\tau}) = \frac{\sqrt{\pi}}{\sqrt{2\pi p\tau}} e^{-\frac{x^2}{4p\tau}},$$

$$\mathcal{F}^{-1}(e^{-pw^2\tau}) = \frac{1}{\sqrt{2p\tau}} e^{-\frac{x^2}{4p\tau}}. \quad (4.6)$$

We can now apply the convolution theorem to (3.34) to retrieve $u(x, \tau)$.

From the convolution theorem for Fourier transforms, we have

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) g_t(x-s) ds, \quad (4.7)$$

where $g_t(x) = \mathcal{F}^{-1}(e^{-pw^2\tau})$.

Substituting (4.6) into (4.7), we have

$$u(x, \tau) = \frac{1}{\sqrt{4\pi p\tau}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4p\tau}} ds. \quad (4.8)$$

Using (3.17b), we have

$$u(x, \tau) = \frac{1}{\sqrt{4\pi p\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4p\tau}} ds. \quad (4.9)$$

Thus (4.9) is the solution of the heat equation.

5. Conclusion

Pricing for solving the problem of risky asset and its derivatives is one of the elements of the study of mathematical finance and option pricing problem is one of the most important content. In this work, the option pricing equation modeled by fractional Brownian motion has been obtained. Using Fourier transform, it is successfully reduced to a one-dimensional heat equation and then a solution obtained by applying the convolution theorem.

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