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# On Generalized co-Narayana Numbers

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**Abstract**

In this paper, we introduce and investigate a new third order recurrence sequence so called generalized co-Narayana sequence and its two special subsequences which are related to generalized Narayana numbers and its two subsequences. There are close interrelations between recurrence equations of and roots of characteristic equations of generalized Narayana and generalized co-Narayana numbers. We present Binet's formulas, generating functions, some identities, Simson's formulas, recurrence properties, sum formulas and matrices related with these sequences.

## 1 Introduction: Generalized Narayana and co-Narayana Numbers

The generalized Tribonacci numbers

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where $W_0, W_1, W_2$ are arbitrary complex (or real) numbers and $r$, $s$ and $t$ are real numbers with $t \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers $n$.

For $r, s, t$ satisfying Eq. (1.1), the generalized co-Tribonacci numbers

$$\{Y_n(Y_0, Y_1, Y_2; -s, -rt, t^2)\}_{n\geq 0}$$

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(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -sY_{n-1} - rtY_{n-2} + t^2Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{1.2}$$

i.e.,

$$Y_n = r_1Y_{n-1} + s_1Y_{n-2} + t_1Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

where $Y_0, Y_1, Y_2$ are arbitrary complex (or real) numbers and $r_1 = -s$, $s_1 = -rt$, $t_1 = t^2$.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} Y_{-n} &= -\frac{-rt}{t^2}Y_{-(n-1)} - \frac{-s}{t^2}Y_{-(n-2)} + \frac{1}{t^2}Y_{-(n-3)} \\ &= -\frac{s_1}{t_1}Y_{-(n-1)} - \frac{r_1}{t_1}Y_{-(n-2)} + \frac{1}{t_1}Y_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.2) holds for all integer $n$. For more information on generalized Tribonacci and co-Tribonacci numbers, see [8].

Note that we can easily use and modify the results given for $r, s, t$ in [8] by substituting $r_1, s_1, t_1$ for $r, s, t$ and we will do this in this paper.

There are close interrelations between roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci numbers, see [8, Lemma 17.]: If $\alpha, \beta, \gamma$ are the roots of characteristic equation of $\{W_n\}$ which is given as

$$z^3 - rz^2 - sz - t = 0,$$

and if $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation of $\{Y_n\}$ which is given as

$$y^3 - r_1y^2 - s_1y - t_1 = y^3 + sy^2 + rty - t^2 = 0,$$

then we get

$$\begin{aligned} \theta_1 &= \beta\gamma, \\ \theta_2 &= \alpha\beta, \\ \theta_3 &= \alpha\gamma. \end{aligned}$$

There are also close connections and relations between recurrence equations of generalized Tribonacci and generalized co-Tribonacci numbers, see, for example, Lemma 32 in [8].

In this paper, we consider the case $r = 1$, $s = 0$, $t = 1$ so that $r_1 = -s = 0$, $s_1 = -rt = -1$, $t_1 = t^2 = 1$.

In the next section, we also use the notation $r = 0, s = -1, t = 1$ for $r_1 = 0$, $s_1 = -1$, $t_1 = 1$, to use results in the paper [8]. Now, in this section, for the case $r = 1$, $s = 0$, $t = 1$ we present some well known results.

The Narayana numbers was introduced by the Indian mathematician Narayana in the 14th century, while studying the problem of a herd of cows and calves, see [1,5] for details. Narayana's cows problem is a problem similar to the Fibonacci's rabbit problem which can be given as follows: A cow produces one calf every year and beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years? This problem can be solved in the same way that Fibonacci solved its problem about rabbits (see [2]). If $n$ is the year, then the Narayana problem can be modelled by the recurrence $N_{n+3} = N_{n+2} + N_n$, with $n \geq 0$, $N_0 = 0, N_1 = 1, N_2 = 1$, see [1]. The first few terms are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28...$, (the sequence A000930 in [6]). This sequence is called Narayana sequence (also called Fibonacci-Narayana sequence or Narayana's cows sequence).

Recently, there has been considerable interest in the Narayana sequence and its generalizations,see for example Soykan [7 and the references given therein].

Note that Narayana sequence (or Fibonacci-Narayana sequence or Narayana's cows sequence) named after a 14th-century Indian mathematician Narayana. In literature, there is a sequence which is also called Narayana sequence (named after Canadian mathematician T. V. Narayana (1930–1987)) and is defined by the numbers (the sequence A001263 in [6])

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

where $1 \leq k \leq n$. These type of Narayana numbers (in fact, a q-analogue of them) were first studied by MacMahon [3, Article 495] and were later rediscovered by Narayana [4]. It is well known that for any positive integer $n$,

$$C_n = \sum_{k=1}^{n} N(n,k)$$

where $C_n$ are Catalan numbers and given by $C_n = \frac{1}{n+1}\binom{2n}{n}$ and satisfies the recurrence $C_{n+1} = \frac{4n+2}{n+2} C_n$ where $C_0 = 1$.

The purpose of this chapter is to study a generalisation of Narayana sequence (Narayana's cows sequence). We define and investigate the generalized Narayana sequence and we deal with, in detail, one special case besides Narayana sequence which we call it Narayana-Lucas sequence.

In this chapter, we consider the case $r = 1$, $s = 0, t = 1$. A generalized Narayana sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-1} + W_{n-3} \tag{1.3}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (1.3) holds for all integer $n$.

Generalized Narayana sequence and its special cases have been studied by many authors, see for example Soykan [7 and the references given therein].

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), its characteristic equation (cubic equation) is

$$z^3 - z^2 - 1 = 0. \tag{1.4}$$

The roots $\alpha, \beta, \gamma$ of characteristic equation of $\{W_n\}$ are given as

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{aligned}
\alpha + \beta + \gamma &= 1, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The sequence $\{W_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of $W_n$, Binet's formula of $W_n$ can be given as follows:

**Theorem 1.** *For all integers $n$, Binet's formula of generalized Narayana numbers is given as follows:*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n,
\end{aligned}$$

*where*

$$\begin{aligned}
p_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \\
p_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \\
p_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0
\end{aligned}$$

*and*

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)} \\ &= \frac{(\alpha W_2+\alpha(-1+\alpha)W_1+W_0)}{\alpha^2+3}, \\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)} \\ &= \frac{(\beta W_2+\beta(-1+\beta)W_1+W_0)}{\beta^2+3}, \\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{(\gamma W_2+\gamma(-1+\gamma)W_1+W_0)}{\gamma^2+3}. \end{aligned}$$

*Proof.* Set $r=1$, $s=0, t=1$ in [8, Theorem 3 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence $W_n$.

**Lemma 2.** *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Narayana numbers* $\{W_n\}_{n\geq 0}$*. Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0+(W_1-W_0)z+(W_2-W_1)z^2}{1-z-z^3}.$$

*Proof.* Set $r=1$, $s=0, t=1$ in [8, Lemma 9.]. □

Two special cases of the sequence $\{W_n\}$ are the well known Narayana sequence $\{N_n\}_{n\geq 0}$ and Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$. Narayana sequence $\{N_n\}_{n\geq 0}$, Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$N_{n+3} = N_{n+2} + N_n, \qquad N_0 = 0, N_1 = 1, N_2 = 1, \tag{1.5}$$

and

$$U_{n+3} = U_{n+2} + U_n, \qquad U_0 = 3, U_1 = 1, U_2 = 1. \tag{1.6}$$

The sequences $\{N_n\}_{n\geq 0}$, $\{U_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$N_{-n} = -N_{-(n-2)} + N_{-(n-3)}$$

and

$$U_{-n} = -U_{-(n-2)} + U_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.5)-(1.6) hold for all integer $n$.

For all integers $n$, Binet's formula of Narayana and Narayana-Lucas numbers (using initial conditions (1.5) and (1.6) in Theorem 1) can be expressed as follows:

**Theorem 3.** *For all integers $n$, Binet's formulas of Narayana and Narayana-Lucas numbers are*

$$\begin{aligned} N_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{\alpha^2+3} + \frac{\beta^{n+2}}{\beta^2+3} + \frac{\gamma^{n+2}}{\gamma^2+3}, \end{aligned}$$

*and*

$$U_n = \alpha^n + \beta^n + \gamma^n,$$

*respectively.*

Lemma 2 gives the following results as particular examples (generating functions of Narayana and Narayana-Lucas numbers).

**Corollary 4.** *Generating functions of Narayana and Narayana-Lucas numbers are*

$$\sum_{n=0}^{\infty} N_n z^n = \frac{z}{1-z-z^3}$$

*and*

$$\sum_{n=0}^{\infty} U_n z^n = \frac{3-2z}{1-z-z^3}$$

*respectively.*

We can give a few basic relations between $\{U_n\}$ and $\{N_n\}$.

**Lemma 5.** *The following equalities are true:*

**(a)** $U_n = 3N_{n+4} - 5N_{n+3} + 2N_{n+2}$.

**(b)** $U_n = -2N_{n+3} + 2N_{n+2} + 3N_{n+1}$.

**(c)** $U_n = 3N_{n+1} - 2N_n$.

**(d)** $U_n = N_n + 3N_{n-2}$.

**(e)** $31N_n = -3U_{n+4} + U_{n+3} + 11U_{n+2}$.

**(f)** $31N_n = -2U_{n+3} + 11U_{n+2} - 3U_{n+1}$.

**(g)** $31N_n = 9U_{n+2} - 3U_{n+1} - 2U_n$.

**(h)** $31N_n = 6U_{n+1} - 2U_n + 9U_{n-1}$.

**(i)** $31N_n = 4U_n + 9U_{n-1} + 6U_{n-2}$.

## 2 Generalized co-Narayana Numbers

If $r = 1$, $s = 0$, $t = 1$, then we get $r_1 = -s = 0$, $s_1 = -rt = -1$, $t_1 = t^2 = 1$. From now on, throughout the paper, we also use the notation $r = 0, s = -1, t = 1$ for $r_1 = 0$, $s_1 = -1$, $t_1 = 1$ and we consider the case $r = 0, s = -1, t = 1$ to use results in the paper [8].

In this section, we define and investigate a new sequence and its two special cases, namely the generalized co-Narayana, co-Narayana and co-Narayana-Lucas numbers. The generalized co-Narayana numbers

$$\{Y_n(Y_0, Y_1, Y_2; 0, -1, 1)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -Y_{n-2} + Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{2.1}$$

where $Y_0, Y_1, Y_2$ are arbitrary complex (or real) numbers with real coefficients.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n} = Y_{-(n-1)} + Y_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2.1) holds for all integer $n$.

The first few generalized co-Narayana numbers with positive subscript and negative subscript are given in the following Table 1.

**Remark 6.** *In this paper we will extensively use the paper [8]. Note that in the notation of [8], here we have* $r = 1$, $s = 0$, $t = 1$ *and* $r_1 = 0$, $s_1 = -1$, $t_1 = 1$. *For simplicity, we can use the result of [8] by taking and replacing* $r = 0, s = -1, t = 1$.

As $\{Y_n\}$ is a third-order recurrence sequence (difference equation), its characteristic equation (cubic equation) is

$$y^3 + y - 1 = 0.$$

Table 1: A few generalized co-Narayana numbers

| $n$ | $Y_n$ | $Y_{-n}$ |
|---|---|---|
| 0 | $Y_0$ | $Y_0$ |
| 1 | $Y_1$ | $Y_0+Y_2$ |
| 2 | $Y_2$ | $Y_0+Y_1+Y_2$ |
| 3 | $Y_0-Y_1$ | $2Y_0+Y_1+Y_2$ |
| 4 | $Y_1-Y_2$ | $3Y_0+Y_1+2Y_2$ |
| 5 | $Y_1-Y_0+Y_2$ | $4Y_0+2Y_1+3Y_2$ |
| 6 | $Y_0-2Y_1+Y_2$ | $6Y_0+3Y_1+4Y_2$ |
| 7 | $Y_0-2Y_2$ | $9Y_0+4Y_1+6Y_2$ |
| 8 | $3Y_1-2Y_0$ | $13Y_0+6Y_1+9Y_2$ |
| 9 | $3Y_2-2Y_1$ | $19Y_0+9Y_1+13Y_2$ |
| 10 | $3Y_0-3Y_1-2Y_2$ | $28Y_0+13Y_1+19Y_2$ |
| 11 | $5Y_1-2Y_0-3Y_2$ | $41Y_0+19Y_1+28Y_2$ |
| 12 | $Y_1-3Y_0+5Y_2$ | $60Y_0+28Y_1+41Y_2$ |
| 13 | $5Y_0-8Y_1+Y_2$ | $88Y_0+41Y_1+60Y_2$ |

The roots $\theta_1,\theta_2,\theta_3$ of characteristic equation of $\{Y_n\}$ are given as

$$\begin{aligned}
\theta_1 &= \left(\frac{1}{2}+\sqrt{\frac{31}{108}}\right)^{1/3}-\left(-\frac{1}{2}+\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\theta_2 &= \omega\left(\frac{1}{2}+\sqrt{\frac{31}{108}}\right)^{1/3}-\omega^2\left(-\frac{1}{2}+\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\theta_3 &= \omega^2\left(\frac{1}{2}+\sqrt{\frac{31}{108}}\right)^{1/3}-\omega\left(-\frac{1}{2}+\sqrt{\frac{31}{108}}\right)^{1/3},
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\left\{\begin{array}{c}
\theta_1+\theta_2+\theta_3=0, \\
\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3=1, \\
\theta_1\theta_2\theta_3=1.
\end{array}\right.$$

Note that there are an important relation between $\theta_1, \theta_2, \theta_3$ and $\alpha, \beta, \gamma$:

$$\theta_1 = \beta\gamma,$$
$$\theta_2 = \alpha\beta,$$
$$\theta_3 = \alpha\gamma.$$

The sequence $\{Y_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of $Y_n$, Binet's formula of $Y_n$ can be given as follows:

**Theorem 7.** *For all integers $n$, Binet's formula of generalized co-Narayana numbers is given as follows:*

$$\begin{aligned} Y_n &= \frac{p_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{p_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{p_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n, \end{aligned}$$

*where*

$$p_1 = Y_2 - (\theta_2+\theta_3)Y_1 + \theta_2\theta_3Y_0,\ p_2 = Y_2 - (\theta_1+\theta_3)Y_1 + \theta_1\theta_3Y_0,\ p_3 = Y_2 - (\theta_1+\theta_2)Y_1 + \theta_1\theta_2Y_0,$$

*and*

$$\begin{aligned} A_1 &= \frac{p_1}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)} \\ &= \frac{(\theta_1Y_2+\theta_1\theta_1Y_1+Y_0)}{-2\theta_1+3}, \\ A_2 &= \frac{p_2}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)} \\ &= \frac{(\theta_2Y_2+\theta_2\theta_2Y_1+Y_0)}{-2\theta_2+3}, \\ A_3 &= \frac{p_3}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{(\theta_3Y_2+\theta_3\theta_3Y_1+Y_0)}{-2\theta_3+3}. \end{aligned}$$

*Proof.* For the proof, take $r = 0$, $s = -1$, $t = 1$ in [8, Theorem 3 (a)] or $r_1 = 0, s_1 = -1, t_1 = 1$ in [8, Theorem 19 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} Y_n z^n$ of the sequence $Y_n$.

**Lemma 8.** *Suppose that $f_{Y_n}(z) = \sum_{n=0}^{\infty} Y_n z^n$ is the ordinary generating function of the generalized co-Narayana numbers $\{Y_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} Y_n z^n$ is given by*

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{Y_0 + Y_1z + (Y_2+Y_0)z^2}{1+z^2-z^3}.$$

Table 2: The first few values of the special third-order numbers with positive and negative subscripts.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $M_n$ | 0 | 1 | 0 | $-1$ | 1 | 1 | $-2$ | 0 | 3 | $-2$ | $-3$ | 5 | 1 | $-8$ |
| $M_{-n}$ | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 41 |
| $S_n$ | 3 | 0 | $-2$ | 3 | 2 | $-5$ | 1 | 7 | $-6$ | $-6$ | 13 | 0 | $-19$ | 13 |
| $S_{-n}$ | 3 | 1 | 1 | 4 | 5 | 6 | 10 | 15 | 21 | 31 | 46 | 67 | 98 | 144 |

*Proof.* Set $r = 0$, $s = -1$, $t = 1$ in [8, Lemma 9] or $r_1 = 0, s_1 = -1, t_1 = 1$ in [8, Lemma 24]. □

In this paper, we define and investigate, in detail, two special cases of the generalized co-Narayana numbers $\{Y_n\}$ which we call them co-Narayana and co-Narayana-Lucas numbers. co-Narayana numbers $\{M_n\}_{n\geq 0}$ and co-Narayana-Lucas numbers $\{S_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$M_{n+3} = -M_{n+1} + M_n, \qquad M_0 = 0, M_1 = 1, M_2 = 0, \tag{2.2}$$

$$S_{n+3} = -S_{n+1} + S_n, \qquad S_0 = 3, S_1 = 0, S_2 = -2. \tag{2.3}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= M_{-(n-1)} + M_{-(n-3)}, \\ S_{-n} &= S_{-(n-1)} + S_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.2) and (2.3) hold for all integers $n$.

Next, we present the first few values of the co-Narayana and co-Narayana-Lucas numbers with positive and negative subscripts.

For all integers $n$, Binet's formula of co-Narayana and co-Narayana-Lucas numbers (using initial conditions (2.2) and (2.3) in Theorem 7) can be expressed as follows:

**Theorem 9.** *For all integers $n$, Binet's formulas of co-Narayana and co-Narayana-Lucas numbers are*

$$\begin{aligned} M_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{\theta_1^{n+2}}{-2\theta_1+3} + \frac{\theta_2^{n+2}}{-2\theta_2+3} + \frac{\theta_3^{n+2}}{-2\theta_3+3}, \end{aligned}$$

*and*

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n,$$

*respectively.*

Lemma 8 gives the following results as particular examples (generating functions of co-Narayana and co-Narayana-Lucas numbers).

**Corollary 10.** *Generating functions of co-Narayana and co-Narayana-Lucas numbers are*

$$\sum_{n=0}^{\infty} M_n z^n = \frac{z}{1+z^2-z^3},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{3+z^2}{1+z^2-z^3},$$

*respectively.*

## 3 Connections between $N_n$, $U_n$ and $M_n, S_n$

$S_n$ can be given as follows.

**Lemma 11.** *For all integers $n$, we have the following formula for $S_n$:*

$$\begin{aligned} S_n &= \theta_1^n + \theta_2^n + \theta_3^n \\ &= \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n. \end{aligned}$$

*Proof.* Use [8, Lemma 30]. □

We can present the relations between $M_n$, $S_n$ and $N_n$, $U_n$ as follows.

**Lemma 12.** *For all integers $n$, we have the following formulas:*

**(a)** $S_n = \frac{1}{2}(U_n^2 - U_{2n})$.

**(b)** $M_n = N_{-n-1}$ *and* $M_{-n} = N_{n-1}$.

**(c)** $S_n = U_{-n}$ *and* $S_{-n} = U_n$.

*Proof.* Use [8, Lemma 32]. □

## 4 Some Identities of Generalized co-Narayana Numbers

In this section, we obtain some identities of generalized co-Narayana, co-Narayana and co-Narayana-Lucas numbers. First, we can give a few basic relations between $\{M_n\}$ and $\{S_n\}$.

**Lemma 13.** *The following equalities are true:*

**(a)** $S_n = 4M_{n+4} + M_{n+3} + 5M_{n+2}$.

**(b)** $S_n = M_{n+3} + M_{n+2} + 4M_{n+1}$.

**(c)** $S_n = M_{n+2} + 3M_{n+1} + M_n$.

**(d)** $S_n = 3M_{n+1} + M_{n-1}$.

**(e)** $S_n = 3M_{n-2} - 2M_{n-1}$.

**(f)** $31M_n = 4S_{n+4} + 6S_{n+3} + 13S_{n+2}$.

**(g)** $31M_n = 6S_{n+3} + 9S_{n+2} + 4S_{n+1}$.

**(h)** $31M_n = 9S_{n+2} - 2S_{n+1} + 6S_n$.

**(i)** $31M_n = -2S_{n+1} - 3S_n + 9S_{n-1}$.

**(j)** $31M_n = -3S_n + 11S_{n-1} - 2S_{n-2}$.

*Proof.* Set $G_n = M_n$, $H_n = S_n$ and $r = 0, s = -1, t = 1$ in [8, Lemma 36]. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we give a few basic relations between $\{M_n\}$ and $\{Y_n\}$.

**Lemma 14.** *The following equalities are true:*

**(a)** $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + Y_0Y_2^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_0Y_1Y_2)M_n = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n$.

**(b)** $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + Y_0Y_2^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_0Y_1Y_2)M_n = (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_n + (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n-1}$.

**(c)** $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + Y_0Y_2^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_0Y_1Y_2)M_n = (Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_n + (-Y_2^2 + Y_0^2 - Y_1Y_2 - Y_0Y_2)Y_{n-1} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n-2}$.

**(d)** $Y_n = (Y_2 + Y_0)M_{n+2} + Y_0M_{n+1} + (Y_2 + Y_1 + Y_0)M_n$.

**(e)** $Y_n = Y_0M_{n+1} + Y_1M_n + (Y_2 + Y_0)M_{n-1}$.

**(f)** $Y_n = Y_1M_n + Y_2M_{n-1} + Y_0M_{n-2}$.

*Proof.* Set $W_n = Y_n$, $G_n = M_n$ and $r = 0$, $s = -1$, $t = 1$ in [8, Lemma 37]. □

Now, we present a few basic relations between $\{S_n\}$ and $\{Y_n\}$.

**Lemma 15.** *The following equalities are true:*

**(a)** $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + Y_0Y_2^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_0Y_1Y_2)S_n = (3Y_2^2 + Y_1^2 + 2Y_0Y_2 - 3Y_0Y_1)Y_{n+2} + (3Y_1^2 + 2Y_1Y_2 - 3Y_0Y_2 - 2Y_0^2 + 2Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_1^2 + 3Y_0^2 - 3Y_1Y_2 - 4Y_0Y_1)Y_n$.

**(b)** $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + Y_0Y_2^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_0Y_1Y_2)S_n = (3Y_1^2 + 2Y_1Y_2 - 3Y_0Y_2 - 2Y_0^2 + 2Y_0Y_1)Y_{n+1} + (3Y_0^2 - 2Y_2^2 - 3Y_1Y_2 - 2Y_0Y_2 - Y_0Y_1)Y_n + (3Y_2^2 + Y_1^2 + 2Y_0Y_2 - 3Y_0Y_1)Y_{n-1}$.

**(c)** $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + Y_0Y_2^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_0Y_1Y_2)S_n = (-2Y_2^2 + 3Y_0^2 - 3Y_1Y_2 - 2Y_0Y_2 - Y_0Y_1)Y_n + (3Y_2^2 - 2Y_1^2 + 2Y_0^2 - 2Y_1Y_2 + 5Y_0Y_2 - 5Y_0Y_1)Y_{n-1} + (3Y_1^2 - 2Y_0^2 + 2Y_1Y_2 - 3Y_0Y_2 + 2Y_0Y_1)Y_{n-2}$.

**(d)** $31Y_n = (6Y_2 + 9Y_1 + 4Y_0)S_{n+2} + (9Y_2 - 2Y_1 + 6Y_0)S_{n+1} + (4Y_2 + 6Y_1 + 13Y_0)S_n$.

**(e)** $31Y_n = (9Y_2 - 2Y_1 + 6Y_0)S_{n+1} + (-2Y_2 - 3Y_1 + 9Y_0)S_n + (6Y_2 + 9Y_1 + 4Y_0)S_{n-1}$.

**(f)** $31Y_n = (-2Y_2 - 3Y_1 + 9Y_0)S_n + (-3Y_2 + 11Y_1 - 2Y_0)S_{n-1} + (9Y_2 - 2Y_1 + 6Y_0)S_{n-2}$.

*Proof.* Set $W_n = Y_n$, $H_n = S_n$, and $r = 0$, $s = -1$, $t = 1$ in [8, Lemma 38]. □

We can present identities between $N_n, U_n$ and $M_n, S_n$ by using Lemmas given above.

**Lemma 16.** *For all integers $n$, we have the following formulas:*

**(a)** $31M_{-n} = 9U_{n+1} - 3U_n - 2U_{n-1}$.

**(b)** $S_{-n} = 3N_{n+1} - 2N_n$.

**(c)** $2S_n = (3N_{n+1} - 2N_n)^2 - (3N_{2n+1} - 2N_{2n})$.

**(d)** $U_{-n} = M_{n+2} + 3M_{n+1} + M_n$.

**(e)** $31N_{-n-1} = 9S_{n+2} - 2S_{n+1} + 6S_n$.

**(f)** $31N_{-n} = 6S_{n+2} + 9S_{n+1} + 4S_n$.

*Proof.* Use Lemmas 5, 12, 13. □

Now, we present some identities of generalized co-Narayana numbers and its special cases.

**Lemma 17.** *Suppose that* $\{X_n\}_{n\geq 0} = \{X_n(X_0, X_1, X_2)\}_{n\geq 0}$ *is also defined by the third-order recurrence relations*

$$X_n = -X_{n-2} + X_{n-3} \tag{4.1}$$

*i.e.,*

$$X_{n+3} = -X_{n+1} + X_n$$

*with the initial values* $X_0, X_1, X_2$ *not all being zero and*

$$X_{-n} = X_{-(n-1)} + X_{-(n-3)}$$

*so that (4.1) is true for all integer* $n$.

*Then the following equalities are true:*

**(a)**

$$(X_0X_3^2 + X_1^2X_4 + X_2^3 - X_0X_2X_4 - 2X_1X_2X_3)Y_n = q_1X_{n+2} + q_2X_{n+1} + q_3X_n$$

*where*

$$q_1 = (X_1^2 - X_0X_2)Y_2 + (X_0X_3 - X_1X_2)Y_1 + (X_2^2 - X_1X_3)Y_0$$
$$q_2 = (X_0X_3 - X_1X_2)Y_2 + (X_2^2 - X_0X_4)Y_1 + (X_1X_4 - X_2X_3)Y_0$$
$$q_3 = (X_2^2 - X_1X_3)Y_2 + (X_1X_4 - X_2X_3)Y_1 + (X_3^2 - X_2X_4)Y_0$$

**(b)**

$$(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)M_n = q_4Y_{n+2} + q_5Y_{n+1} + q_6Y_n$$

*where*

$$q_4 = Y_0^2 - Y_1Y_2 - Y_0Y_1$$
$$q_5 = Y_2^2 + Y_0Y_2 - Y_0Y_1$$
$$q_6 = Y_1^2 - Y_0Y_2$$

**(c)**

$$Y_n = q_7M_{n+2} + q_8M_{n+1} + q_9M_n$$

*where*

$$q_7 = Y_2 + Y_0$$
$$q_8 = Y_0$$
$$q_9 = Y_2 + Y_1 + Y_0$$

**(d)**

$$(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)S_n = q_{10}Y_{n+2} + q_{11}Y_{n+1} + q_{12}Y_n$$

*where*

$q_{10} = 3Y_2^2 + Y_1^2 + 2Y_0Y_2 - 3Y_0Y_1$

$q_{11} = 3Y_1^2 - 2Y_0^2 + 2Y_1Y_2 - 3Y_0Y_2 + 2Y_0Y_1$

$q_{12} = Y_2^2 + Y_1^2 + 3Y_0^2 - 3Y_1Y_2 - 4Y_0Y_1$

**(e)**

$$31Y_n = q_{13}S_{n+2} + q_{14}S_{n+1} + q_{15}S_n$$

*where*

$q_{13} = 6Y_2 + 9Y_1 + 4Y_0$

$q_{14} = 9Y_2 - 2Y_1 + 6Y_0$

$q_{15} = 4Y_2 + 6Y_1 + 13Y_0$

*Proof.*

**(a)** Writing

$$Y_n = q_1 \times X_{n+2} + q_2 \times X_{n+1} + q_3 \times X_n$$

and solving the system of equations

$$\begin{aligned} Y_0 &= q_1 \times X_2 + q_2 \times X_1 + q_3 \times X_0 \\ Y_1 &= q_1 \times X_3 + q_2 \times X_2 + q_3 \times X_1 \\ Y_2 &= q_1 \times X_4 + q_2 \times X_3 + q_3 \times X_2 \end{aligned}$$

we find the required identity.

**(b)** Replace $Y_n$ and $X_n$ with $M_n$ and $Y_n$, respectively in (a).

**(c)** Replace $X_n$ with $M_n$ in (a).

**(d)** Replace $Y_n$ and $X_n$ with $S_n$ and $Y_n$, respectively in (a).

**(e)** Replace $X_n$ with $S_n$ in (a). □

## 5 Simson's Formulas of co-Narayana Numbers

The following theorem gives Simson's formula of the generalized co-Narayana numbers $\{Y_n\}$.

**Theorem 18** (Simson's Formula of Generalized co-Narayana Numbers). *For all integers $n$, we have*

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_2+Y_0 \\ Y_0 & Y_2+Y_0 & Y_2+Y_1+Y_0 \end{vmatrix}.$$

*Proof.* Set $W_n = Y_n$ and $r = 0$, $s = -1$, $t = 1$ in [8, Theorem 33]. □

The previous theorem gives the following results as particular examples.

**Corollary 19.** *For all integers $n$, Simson's formula of co-Narayana and co-Narayana-Lucas numbers are given as*

$$\begin{vmatrix} M_{n+2} & M_{n+1} & M_n \\ M_{n+1} & M_n & M_{n-1} \\ M_n & M_{n-1} & M_{n-2} \end{vmatrix} = -1,$$

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -31,$$

*respectively.*

*Proof.* Set $Y_n = M_n$ and $Y_n = S_n$ in Theorem 18, respectively. □

## 6 Recurrence Properties of Generalized co-Narayana Numbers

The generalized co-Narayana numbers $Y_n$ at negative indices can be expressed by the sequence itself at positive indices.

**Theorem 20.** *For $n \in \mathbb{Z}$, we have*

$$Y_{-n} = Y_{2n} - S_n Y_n + \frac{1}{2}(S_n^2 - S_{2n})Y_0.$$

*Proof.* Set $W_n = Y_n$, $U_n = S_n$ and $r = 0$, $s = -1$, $t = 1$ in [8, Theorem 39]. □

As special cases of the above Theorem, we have the following Corollary.

**Corollary 21.** *For $n \in \mathbb{Z}$, we have*

**(a)**
$$M_{-n} = -M_n^2 + M_{2n} - M_{n+2}M_n - 3M_{n+1}M_n.$$

**(b)**
$$S_{-n} = \frac{1}{2}(S_n^2 - S_{2n}).$$

*Proof.* Take $r = 0$, $s = -1$, $t = 1$, and $N_n = M_n$ and $U_n = S_n$, respectively, in [8, Corollary 42] or set $Y_n = M_n$ and $Y_n = S_n$, respectively, in Theorem 20. □

The last Corollary can be written in the following form by using Lemma 12.

**Corollary 22.** *For $n \in \mathbb{Z}$, we have*

**(a)**
$$N_{n-1} = -M_n^2 + M_{2n} - M_{n+2}M_n - 3M_{n+1}M_n.$$

**(b)**
$$U_n = \frac{1}{2}(S_n^2 - S_{2n}).$$

*Proof.* Use Lemma 12 and Corollary 22. □

## 7 Sum Formulas $\sum_{k=0}^{n} Y_k$, $\sum_{k=0}^{n} Y_{2k}$, $\sum_{k=0}^{n} Y_{2k+1}$, $\sum_{k=0}^{n} Y_{-k}$, $\sum_{k=0}^{n} Y_{-2k}$, $\sum_{k=0}^{n} Y_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{2n} z^n$, $\sum_{n=0}^{\infty} Y_{2n+1} z^n$, $\sum_{n=0}^{\infty} Y_{-n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n+1} z^n$ of Generalized co-Narayana Numbers

Next, we present sum formulas of generalized co-Narayana numbers

**Theorem 23.** *For $n \geq 0$, we have the following sum formulas for generalized co-Narayana numbers:*

**(a)** $\sum\limits_{k=0}^{n} Y_k = -Y_{n+2} - Y_{n+1} - Y_n + Y_2 + Y_1 + 2Y_0.$

**(b)** $\sum\limits_{k=0}^{n} Y_{2k} = \frac{1}{3}(-2Y_{2n+2} - Y_{2n+1} - Y_{2n} + 2Y_2 + Y_1 + 4Y_0).$

**(c)** $\sum\limits_{k=0}^{n} Y_{2k+1} = \frac{1}{3}(-Y_{2n+2} + Y_{2n+1} - 2Y_{2n} + Y_2 + 2Y_1 + 2Y_0).$

**(d)** $\sum_{k=0}^{n} Y_{-k} = Y_{-n+2} + Y_{-n+1} + 2Y_{-n} - Y_2 - Y_1 - Y_0.$

**(e)** $\sum_{k=0}^{n} Y_{-2k} = \frac{1}{3}(Y_{-2n-1} + 2Y_{-2n} + Y_{-2n-2} - 2Y_2 - Y_1 - Y_0).$

**(f)** $\sum_{k=0}^{n} Y_{-2k+1} = \frac{1}{3}(Y_{-2n} - Y_{-2n-1} + 2Y_{-2n-2} - Y_2 + Y_1 - 2Y_0).$

*Proof.*

**(a)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [8, Theorem 62 (a) (i)].

**(b)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [8, Theorem 62 (b) (i)].

**(c)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [8, Theorem 62 (c) (i)].

**(d)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [8, Theorem 62 (d) (i)].

**(e)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [8, Theorem 62 (e) (i)].

**(f)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [8, Theorem 62 (f) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Narayana numbers (take $Y_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 0$).

**Corollary 24.** *For $n \geq 0$, co-Narayana numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} M_k = -M_{n+2} - M_{n+1} - M_n + 1.$

**(b)** $\sum_{k=0}^{n} M_{2k} = \frac{1}{3}(-2M_{2n+2} - M_{2n+1} - M_{2n} + 1).$

**(c)** $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{3}(-M_{2n+2} + M_{2n+1} - 2M_{2n} + 2).$

**(d)** $\sum_{k=0}^{n} M_{-k} = M_{-n+2} + M_{-n+1} + 2M_{-n} - 1.$

**(e)** $\sum_{k=0}^{n} M_{-2k} = \frac{1}{3}(M_{-2n-1} + 2M_{-2n} + M_{-2n-2} - 1).$

**(f)** $\sum_{k=0}^{n} M_{-2k+1} = \frac{1}{3}(M_{-2n} - M_{-2n-1} + 2M_{-2n-2} + 1).$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -2$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Narayana-Lucas numbers.

**Corollary 25.** *For $n \geq 0$, co-Narayana-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} S_k = -S_{n+2} - S_{n+1} - S_n + 4.$

**(b)** $\sum\limits_{k=0}^{n} S_{2k} = \frac{1}{3}(-2S_{2n+2} - S_{2n+1} - S_{2n} + 8).$

**(c)** $\sum\limits_{k=0}^{n} S_{2k+1} = \frac{1}{3}(-S_{2n+2} + S_{2n+1} - 2S_{2n} + 4).$

**(d)** $\sum\limits_{k=0}^{n} S_{-k} = S_{-n+2} + S_{-n+1} + 2S_{-n} - 1.$

**(e)** $\sum\limits_{k=0}^{n} S_{-2k} = \frac{1}{3}(S_{-2n-1} + 2S_{-2n} + S_{-2n-2} + 1).$

**(f)** $\sum\limits_{k=0}^{n} S_{-2k+1} = \frac{1}{3}(S_{-2n} - S_{-2n-1} + 2S_{-2n-2} - 4).$

Next, we give the ordinary generating function of special cases of the generalized co-Narayana numbers $\{Y_{mn+j}\}$.

**Corollary 26.** *The ordinary generating functions of the sequences $Y_n$, $Y_{2n}$, $Y_{2n+1}$, $Y_{-n}$, $Y_{-2n}$, $Y_{-2n+1}$ are given as follows:*

**(a)** $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.826031).$

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{(Y_0 + Y_2)z^2 + Y_1 z + Y_0}{-z^3 + z^2 + 1}.$$

**(b)** $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.682327).$

$$\sum_{n=0}^{\infty} Y_{2n} z^n = \frac{(Y_0 + Y_1 + Y_2)z^2 + (2Y_0 + Y_2)z + Y_0}{-z^3 + z^2 + 2z + 1}.$$

**(c)** $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.682327).$

$$\sum_{n=0}^{\infty} Y_{2n+1} z^n = \frac{(Y_0 + Y_2)z^2 + (Y_0 + Y_1)z + Y_1}{-z^3 + z^2 + 2z + 1}.$$

**(d)** ($|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 0.682327$ ).

$$\sum_{n=0}^{\infty} Y_{-n} z^n = \frac{-Y_1 z^2 - Y_2 z - Y_0}{z^3 + z - 1}.$$

**(e)** ($|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.465571$ ).

$$\sum_{n=0}^{\infty} Y_{-2n} z^n = \frac{-Y_2 z^2 - (Y_1 + Y_2) z - Y_0}{z^3 + 2z^2 + z - 1}.$$

**(f)** ($|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.465571$ ).

$$\sum_{n=0}^{\infty} Y_{-2n+1} z^n = \frac{(Y_1 - Y_0) z^2 - (Y_2 - Y_1 + Y_0) z - Y_1}{z^3 + 2z^2 + z - 1}.$$

*Proof.* $W_n = Y_n$ and $r = 0, s = -1, t = 1$ in [8, Corollary 67.]. □

Now, we consider special cases of the last corollary.

**Corollary 27.** *The ordinary generating functions of special cases of the generalized co-Narayana numbers are given as follows:*

**(a)** ($|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.826031$).

$$\begin{aligned} \sum_{n=0}^{\infty} M_n z^n &= \frac{z}{-z^3 + z^2 + 1}, \\ \sum_{n=0}^{\infty} S_n z^n &= \frac{z^2 + 3}{-z^3 + z^2 + 1}. \end{aligned}$$

**(b)** ($|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.682327$).

$$\begin{aligned} \sum_{n=0}^{\infty} M_{2n} z^n &= \frac{z^2}{-z^3 + z^2 + 2z + 1}, \\ \sum_{n=0}^{\infty} S_{2n} z^n &= \frac{z^2 + 4z + 3}{-z^3 + z^2 + 2z + 1}. \end{aligned}$$

**(c)** ($|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.682327$).

$$\begin{aligned} \sum_{n=0}^{\infty} M_{2n+1} z^n &= \frac{z + 1}{-z^3 + z^2 + 2z + 1}, \\ \sum_{n=0}^{\infty} S_{2n+1} z^n &= \frac{z^2 + 3z}{-z^3 + z^2 + 2z + 1}. \end{aligned}$$

**(d)** $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 0.682327)$.

$$\sum_{n=0}^{\infty} M_{-n}z^n = \frac{-z^2}{z^3+z-1},$$
$$\sum_{n=0}^{\infty} S_{-n}z^n = \frac{2z-3}{z^3+z-1}.$$

**(e)** $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.465571)$.

$$\sum_{n=0}^{\infty} M_{-2n}z^n = \frac{-z}{z^3+2z^2+z-1},$$
$$\sum_{n=0}^{\infty} S_{-2n}z^n = \frac{2z^2+2z-3}{z^3+2z^2+z-1}.$$

**(f)** $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.465571)$.

$$\sum_{n=0}^{\infty} M_{-2n+1}z^n = \frac{z^2+z-1}{z^3+2z^2+z-1},$$
$$\sum_{n=0}^{\infty} S_{-2n+1}z^n = \frac{-3z^2-z}{z^3+2z^2+z-1}.$$

From the last corollary, we obtain the following results for special cases of $z$.

**Corollary 28.** *We have the following infinite sums .*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{M_n}{2^n} = \frac{4}{9},$$
$$\sum_{n=0}^{\infty} \frac{S_n}{2^n} = \frac{26}{9}.$$

**(b)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{M_{2n}}{2^n} = \frac{2}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{2n}}{2^n} = \frac{42}{17}.$$

**(c)** $z=\frac{1}{2}$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{M_{2n+1}}{2^{n}} &= \frac{12}{17},\\ \sum_{n=0}^{\infty}\frac{S_{2n+1}}{2^{n}} &= \frac{14}{17}\end{aligned}$$

**(d)** $z=\frac{1}{2}$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{M_{-n}}{2^{n}} &= \frac{2}{3},\\ \sum_{n=0}^{\infty}\frac{S_{-n}}{2^{n}} &= \frac{16}{3}.\end{aligned}$$

**(e)** $z=\frac{1}{4}$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{M_{-2n}}{4^{n}} &= \frac{16}{39},\\ \sum_{n=0}^{\infty}\frac{S_{-2n}}{4^{n}} &= \frac{152}{39}.\end{aligned}$$

**(f)** $z=\frac{1}{4}$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{M_{-2n+1}}{4^{n}} &= \frac{44}{39},\\ \sum_{n=0}^{\infty}\frac{S_{-2n+1}}{4^{n}} &= \frac{28}{39}.\end{aligned}$$

## 8 Sum Formulas $\sum_{k=0}^{n} z^k Y_k^2$, $\sum_{k=0}^{n} z^k Y_{k+1}Y_k$, $\sum_{k=0}^{n} z^k Y_{k+2}Y_k$ and Generating Functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of Generalized co-Narayana Numbers

Next, we present sum formulas of generalized co-Narayana Numbers numbers.

**Theorem 29.** *For $n\geq 0$, we have the following sum formulas for generalized co-Narayana numbers:*

**(a)** $\sum_{k=0}^{n} Y_k^2 = \frac{1}{3}(Y_{n+3}^2 + Y_{n+2}^2 - 2Y_{n+1}^2 - 2Y_{n+2}Y_{n+3} - 2Y_{n+1}Y_{n+3} - 4Y_{n+1}Y_{n+2} - Y_2^2 - Y_1^2 + 2Y_0^2 + 2Y_1Y_2 + 2Y_0Y_2 + 4Y_0Y_1)$.

**(b)** $\sum_{k=0}^{n} Y_{k+1}Y_k = \frac{1}{3}(-Y_{n+3}^2 - Y_{n+2}^2 - Y_{n+1}^2 - Y_{n+2}Y_{n+3} - Y_{n+1}Y_{n+3} - 2Y_{n+1}Y_{n+2} + Y_2^2 + Y_1^2 + Y_0^2 + Y_1Y_2 + Y_0Y_2 + 2Y_0Y_1)$.

**(c)** $\sum_{k=0}^{n} Y_{k+2}Y_k = \frac{1}{3}(-2Y_{n+3}^2 - 2Y_{n+2}^2 - 2Y_{n+1}^2 + Y_{n+2}Y_{n+3} - 2Y_{n+1}Y_{n+3} + 2Y_{n+1}Y_{n+2} + 2Y_2^2 + 2Y_1^2 + 2Y_0^2 - Y_1Y_2 + 2Y_0Y_2 - 2Y_0Y_1)$.

*Proof.* Note that characteristic equation of the third-order recurrence sequence $Y_n$ is the cubic equation $y^3 + y - 1 = 0$ whose roots are $\theta_1, \theta_2, \theta_3$ with $\theta_1 \neq \theta_2 \neq \theta_3$. In [10, Theorem 2.1]), for $r = 0, s = -1, t = 1$, we get

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) \\ &= (-z^3 + z^2 + 2z + 1)(z^3 + z - 1)\end{aligned}$$

and $\Gamma(1) \neq 0$.

**(a)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [10, Theorem 2.1 (a) (i)] or in [9, Theorem 2.1 (a) (i)].

**(b)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [10, Theorem 2.1 (b) (i)] or in [9, Theorem 2.1 (b) (i)].

**(c)** Set $W_n = Y_n$, $r = 0, s = -1, t = 1$ and $z = 1$ in [10, Theorem 2.1 (c) (i)] or in [9, Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Narayana numbers (take $Y_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 0$).

**Corollary 30.** *For $n \geq 0$, co-Narayana numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(M_{n+3}^2 + M_{n+2}^2 - 2M_{n+1}^2 - 2M_{n+2}M_{n+3} - 2M_{n+1}M_{n+3} - 4M_{n+1}M_{n+2} - 1)$.

**(b)** $\sum_{k=0}^{n} M_{k+1}M_k = \frac{1}{3}(-M_{n+3}^2 - M_{n+2}^2 - M_{n+1}^2 - M_{n+2}M_{n+3} - M_{n+1}M_{n+3} - 2M_{n+1}M_{n+2} + 1)$.

**(c)** $\sum_{k=0}^{n} M_{k+2}M_k = \frac{1}{3}(-2M_{n+3}^2 - 2M_{n+2}^2 - 2M_{n+1}^2 + M_{n+2}M_{n+3} - 2M_{n+1}M_{n+3} + 2M_{n+1}M_{n+2} + 2)$.

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -2$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Narayana-Lucas numbers.

**Corollary 31.** *For $n \geq 0$, co-Narayana-Lucas numbers have the following properties:*

**(a)** $\sum_{k=0}^{n} S_k^2 = \frac{1}{3}(S_{n+3}^2 + S_{n+2}^2 - 2S_{n+1}^2 - 2S_{n+2}S_{n+3} - 2S_{n+1}S_{n+3} - 4S_{n+1}S_{n+2} + 2).$

**(b)** $\sum_{k=0}^{n} S_{k+1}S_k = \frac{1}{3}(-S_{n+3}^2 - S_{n+2}^2 - S_{n+1}^2 - S_{n+2}S_{n+3} - S_{n+1}S_{n+3} - 2S_{n+1}S_{n+2} + 7).$

**(c)** $\sum_{k=0}^{n} S_{k+2}S_k = \frac{1}{3}(-2S_{n+3}^2 - 2S_{n+2}^2 - 2S_{n+1}^2 + S_{n+2}S_{n+3} - 2S_{n+1}S_{n+3} + 2S_{n+1}S_{n+2} + 14).$

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of the sequences $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$.

**Theorem 32.** *Assume that* $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.682327$. *Then the ordinary generating functions of the sequences* $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$ *are given as follows:*

**(a)** $\sum_{n=0}^{\infty} Y_n^2 z^n = \frac{1}{(z^3+z-1)(-z^3+z^2+2z+1)}((Y_0+Y_2)^2 z^5 + (Y_1^2 + 2Y_0Y_1 + 2Y_1Y_2)z^4 + (-Y_2^2 + 2Y_0^2 + 2Y_1Y_0)z^3 - (Y_2^2 + Y_1^2 - Y_0^2)z^2 - (Y_0^2 + Y_1^2)z - Y_0^2).$

**(b)** $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n = \frac{1}{(z^3+z-1)(-z^3+z^2+2z+1)}(Y_0(Y_0+Y_2)z^5 + (Y_0+Y_2)(Y_1+Y_2)z^4 + (Y_1^2 + 2Y_0Y_1 + Y_1Y_2)z^3 + (Y_0Y_1 - Y_0Y_2)z^2 - Y_1(Y_0+Y_2)z - Y_0Y_1).$

**(c)** $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n = \frac{1}{(z^3+z-1)(-z^3+z^2+2z+1)}(Y_1(Y_0+Y_2)z^5 + (Y_0^2 + Y_2Y_0 - Y_1Y_2)z^4 + (Y_0^2 + 2Y_0Y_2 - Y_1Y_0 + Y_2^2)z^3 + (Y_1^2 - Y_1Y_2 - Y_0Y_1 + Y_2^2 + Y_0Y_2)z^2 + (Y_1^2 - Y_0Y_1 - Y_0Y_2)z - Y_0Y_2).$

*Proof.* Set $W_n = Y_n$ and $r = 0, s = -1, t = 1$ in [10, Theorem 3.1] or in [9, Theorem 3.1]. □

Now, we consider special cases of the last Theorem.

**Corollary 33.** *Assume that* $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.682327$. *The ordinary generating functions of the sequences* $\{M_n^2\}$, $\{M_{n+1}M_n\}$, $\{M_{n+2}M_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

**(a)**

$$\sum_{n=0}^{\infty} M_n^2 z^n = \frac{z^4 - z^2 - z}{(z^3+z-1)(-z^3+z^2+2z+1)},$$
$$\sum_{n=0}^{\infty} S_n^2 z^n = \frac{z^5 + 14z^3 + 5z^2 - 9z - 9}{(z^3+z-1)(-z^3+z^2+2z+1)}.$$

**(b)**

$$\sum_{n=0}^{\infty} M_{n+1}M_n z^n = \frac{z^3}{(z^3+z-1)(-z^3+z^2+2z+1)},$$
$$\sum_{n=0}^{\infty} S_{n+1}S_n z^n = \frac{3z^5 - 2z^4 + 6z^2}{(z^3+z-1)(-z^3+z^2+2z+1)}.$$

**(c)**

$$\sum_{n=0}^{\infty} M_{n+2}M_n z^n = \frac{z^2 + z}{(z^3+z-1)(-z^3+z^2+2z+1)},$$
$$\sum_{n=0}^{\infty} S_{n+2}S_n z^n = \frac{3z^4 + z^3 - 2z^2 + 6z + 6}{(z^3+z-1)(-z^3+z^2+2z+1)}.$$

From the last corollary, we obtain the following results for special cases of $z$.

**Corollary 34.** *Some infinite sums of* $\{M_n^2\}$, $\{M_{n+1}M_n\}$, $\{M_{n+2}M_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

**(a)** $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{M_n^2}{2^n} = \frac{44}{51},$$
$$\sum_{n=0}^{\infty} \frac{S_n^2}{2^n} = \frac{670}{51}.$$

**(b)** $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{M_{n+1}M_n}{2^n} = -\frac{8}{51},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+1}S_n}{2^n} = -\frac{94}{51}.$$

**(c)** $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{M_{n+2}M_{n}}{2^{n}} &= -\frac{16}{17},\\ \sum_{n=0}^{\infty}\frac{S_{n+2}S_{n}}{2^{n}} &= -\frac{188}{17}.\end{aligned}$$

## 9 Generalized co-Narayana Numbers by Matrix Methods

In this section, we present matrix representations of the sequences $Y_n, M_n$ and $S_n$. We also introduce Simson matrix and investigate its properties.

### 9.1 Matrix Representations of the Sequences $Y_n, M_n$ and $S_n$

We define the square matrix $A$ of order 3 as:

$$A=\begin{pmatrix} 0 & -1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

such that $\det A=1$. Some properties of matrix $A^n$ can be given as

$$\begin{aligned}A^{n} &= -A^{n-2}+A^{n-3},\\ A^{n+m} &= A^{n}A^{m}=A^{m}A^{n},\end{aligned}$$

for all integers $m$ and $n$. Note that we have the following formulas:

$$\begin{pmatrix} Y_{n+2}\\ Y_{n+1}\\ Y_{n}\end{pmatrix}=\begin{pmatrix} 0 & -1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix} Y_{n+1}\\ Y_{n}\\ Y_{n-1}\end{pmatrix},$$

and

$$\begin{pmatrix} Y_{n+2}\\ Y_{n+1}\\ Y_{n}\end{pmatrix}=\begin{pmatrix} 0 & -1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}^{n}\begin{pmatrix} Y_{2}\\ Y_{1}\\ Y_{0}\end{pmatrix},$$

and

$$\begin{pmatrix} M_{n+2}\\ M_{n+1}\\ M_{n}\end{pmatrix}=\begin{pmatrix} 0 & -1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix} M_{n+1}\\ M_{n}\\ M_{n-1}\end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} M_{n+1} & -M_n + M_{n-1} & M_n \\ M_n & -M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & -M_{n-2} + M_{n-3} & M_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} Y_{n+1} & -Y_n + Y_{n-1} & Y_n \\ Y_n & -Y_{n-1} + Y_{n-2} & Y_{n-1} \\ Y_{n-1} & -Y_{n-2} + Y_{n-3} & Y_{n-2} \end{pmatrix}.$$

**Theorem 35.** *For all integers $m, n$, we have the following properties:*

**(a)** $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} M_{n+1} & -M_n + M_{n-1} & M_n \\ M_n & -M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & -M_{n-2} + M_{n-3} & M_{n-2} \end{pmatrix}.$$

**(b)** $D_1 A^n = A^n D_1$.

**(c)** $D_{n+m} = D_n B_m = B_m D_n$, *i.e.,*

$$\begin{aligned} &\begin{pmatrix} Y_{n+m+1} & -Y_{n+m} + Y_{n+m-1} & Y_{n+m} \\ Y_{n+m} & -Y_{n+m-1} + Y_{n+m-2} & Y_{n+m-1} \\ Y_{n+m-1} & -Y_{n+m-2} + Y_{n+m-3} & Y_{n+m-2} \end{pmatrix} \\ = &\begin{pmatrix} Y_{n+1} & -Y_n + Y_{n-1} & Y_n \\ Y_n & -Y_{n-1} + Y_{n-2} & Y_{n-1} \\ Y_{n-1} & -Y_{n-2} + Y_{n-3} & Y_{n-2} \end{pmatrix} \begin{pmatrix} M_{m+1} & -M_m + M_{m-1} & M_m \\ M_m & -M_{m-1} + M_{m-2} & M_{m-1} \\ M_{m-1} & -M_{m-2} + M_{m-3} & M_{m-2} \end{pmatrix} \\ = &\begin{pmatrix} M_{m+1} & -M_m + M_{m-1} & M_m \\ M_m & -M_{m-1} + M_{m-2} & M_{m-1} \\ M_{m-1} & -M_{m-2} + M_{m-3} & M_{m-2} \end{pmatrix} \begin{pmatrix} Y_{n+1} & -Y_n + Y_{n-1} & Y_n \\ Y_n & -Y_{n-1} + Y_{n-2} & Y_{n-1} \\ Y_{n-1} & -Y_{n-2} + Y_{n-3} & Y_{n-2} \end{pmatrix}. \end{aligned}$$

**(d)**

$$A^n = M_{n-1}A^2 + (-M_{n-2} + M_{n-3})A + M_{n-2}I$$

*i.e.,*

$$A^n = (M_{n+2} + M_n)A^2 + M_n A + (M_{n+2} + M_{n+1} + M_n)I$$

*that is,*

$$A^n = M_{n+2}(A^2 + I) + M_{n+1}I + M_n(A^2 + A + I)$$

*where*

$$I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Set $W_n = Y_n, r = 1,\ s = 0,\ t = 1$ and $G_n = M_n$ in [8, Theorem 51]. □

Next, we present matrix formulas for the generalized co-Narayana and co-Narayana-Lucas numbers.

**Corollary 36.** *For all integers n, we have the following formulas for generalized co-Narayana numbers and co-Narayana-Lucas numbers.*

**(a)** *Generalized co-Narayana numbers.*

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_Y(0)}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

*where*

$a_{11} = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+3} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n+2} + (Y_1^2 - Y_0Y_2)Y_{n+1},$

$a_{21} = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n,$

$a_{31} = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1},$

$a_{12} = -((Y_0^2-Y_1Y_2-Y_0Y_1)Y_{n+2}+(Y_2^2+Y_0Y_2-Y_0Y_1)Y_{n+1}+(Y_1^2-Y_0Y_2)Y_n)+((Y_0^2-Y_1Y_2-Y_0Y_1)Y_{n+1}+(Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1}),$

$a_{22} = -((Y_0^2-Y_1Y_2-Y_0Y_1)Y_{n+1}+(Y_2^2+Y_0Y_2-Y_0Y_1)Y_n+(Y_1^2-Y_0Y_2)Y_{n-1})+((Y_0^2-Y_1Y_2-Y_0Y_1)Y_n+(Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2}),$

$a_{32} = -((Y_0^2-Y_1Y_2-Y_0Y_1)Y_n+(Y_2^2+Y_0Y_2-Y_0Y_1)Y_{n-1}+(Y_1^2-Y_0Y_2)Y_{n-2})+((Y_0^2-Y_1Y_2-Y_0Y_1)Y_{n-1}+(Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n-2} + (Y_1^2 - Y_0Y_2)Y_{n-3}),$

$a_{13} = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n,$

$a_{23} = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1},$

$a_{33} = (Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_n + (Y_2^2 + Y_0Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2},$

*and*

$\Lambda_Y(0) = Y_2^3 + Y_1^3 + Y_0^3 + Y_0Y_2^2 + Y_2Y_1^2 + Y_0Y_1^2 - 2Y_0^2Y_1 - 3Y_2Y_1Y_0.$

**(b)** *co-Narayana-Lucas numbers.*

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{31}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

*where*

$b_{11} = 9S_{n+3} - 2S_{n+2} + 6S_{n+1},$

$b_{21} = 9S_{n+2} - 2S_{n+1} + 6S_n,$

$b_{31} = 9S_{n+1} - 2S_n + 6S_{n-1},$

$b_{12} = -9S_{n+2} + 11S_{n+1} - 8S_n + 6S_{n-1},$

$b_{22} = -9S_{n+1} + 11S_n - 8S_{n-1} + 6S_{n-2},$

$b_{32} = -9S_n + 11S_{n-1} - 8S_{n-2} + 6S_{n-3},$

$b_{13} = 9S_{n+2} - 2S_{n+1} + 6S_n,$

$b_{23} = 9S_{n+1} - 2S_n + 6S_{n-1},$

$b_{33} = 9S_n - 2S_{n-1} + 6S_{n-2}.$

*Proof.* Set $W_n = Y_n, r = 0,\ s = -1,\ t = 1$ and then take $Y_n = S_n$ in [8, Corollary 52]. □

Note that, $a_{12}, a_{22}, a_{32}$ and $b_{12}, b_{22}, b_{32}$ can be written in the following form:

$a_{12} = (-Y_2^2 + Y_0^2 - Y_1Y_2 - Y_0Y_2)Y_{n+1} + (Y_2^2 - Y_1^2 + Y_0^2 - Y_1Y_2 + 2Y_0Y_2 - 2Y_0Y_1)Y_n + (Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_{n-1},$

$a_{22} = (-Y_2^2 + Y_0^2 - Y_1Y_2 - Y_0Y_2)Y_n + (Y_2^2 - Y_1^2 + Y_0^2 - Y_1Y_2 + 2Y_0Y_2 - 2Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_{n-2},$

$a_{32} = (-Y_2^2 + Y_0^2 - Y_1Y_2 - Y_0Y_2)Y_{n-1} + (Y_2^2 - Y_1^2 + Y_0^2 - Y_1Y_2 + 2Y_0Y_2 - 2Y_0Y_1)Y_{n-2} + (Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_{n-3},$

and

$b_{12} = 11S_{n+1} + S_n - 3S_{n-1},$

$b_{22} = 11S_n + S_{n-1} - 3S_{n-2},$

$b_{32} = 11S_{n-1} + S_{n-2} - 3S_{n-3}.$

Now, we present an identity for $Y_{n+m}$.

**Theorem 37.** *(Honsberger's Identity) For all integers $m$ and $n$, we have*

$$\begin{aligned} Y_{n+m} &= Y_nM_{m+1} + Y_{n-1}(-M_m + M_{m-1}) + Y_{n-2}M_m, \\ &= Y_nM_{m+1} + (-Y_{n-1} + Y_{n-2})\,M_m + Y_{n-1}M_{m-1}. \end{aligned}$$

*Proof.* Set $W_n = Y_n, r = 0, s = -1, t = 1$ and then $N_n = M_n$ in [8, Theorem 53]. □

As special cases of the last Theorem, we have the following corollary.

**Corollary 38.** *For all integers $m, n$, we have the following properties:*

$$\begin{aligned} M_{n+m} &= M_n M_{m+1} + M_{n-1}(-M_m + M_{m-1}) + M_{n-2} M_m, \\ S_{n+m} &= S_n M_{m+1} + S_{n-1}(-M_m + M_{m-1}) + S_{n-2} M_m. \end{aligned}$$

Next, we present identies for $Y_{mn+j}$ and its special cases.

**Corollary 39.** *For all integers $m, n, j$, we have the following properties:*

$$\begin{aligned} Y_{mn+j} &= M_{mn-1} Y_{j+2} + (-M_{mn-2} + M_{mn-3}) Y_{j+1} + M_{mn-2} Y_j, \\ M_{mn+j} &= M_{mn-1} M_{j+2} + (-M_{mn-2} + M_{mn-3}) M_{j+1} + M_{mn-2} M_j, \\ S_{mn+j} &= M_{mn-1} S_{j+2} + (-M_{mn-2} + M_{mn-3}) S_{j+1} + M_{mn-2} S_j. \end{aligned}$$

*Proof.* Set $r = 0, s = -1, t = 1$ and $W_n = Y_n$, then take $Y_n = M_n$, $Y_n = S_n$, respectively, in [8, Corollary 55]. □

## 9.2 Simson Matrix and its Properties

For $n \in \mathbb{Z}$, we define

$$f_Y(n) = \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence $Y_n$. Similarly, as special cases of $Y_n$, Simson matrices of the sequences $M_n$ and $S_n$ are

$$f_M(n) = \begin{pmatrix} M_{n+2} & M_{n+1} & M_n \\ M_{n+1} & M_n & M_{n-1} \\ M_n & M_{n-1} & M_{n-2} \end{pmatrix} \text{ and } f_S(n) = \begin{pmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{pmatrix},$$

respectively.

**Lemma 40.** *For all integers $n, m$ and $j$, the followings hold.*

**(a)** $f_Y(n) = -f_Y(n-2) + f_Y(n-3)$.

**(b)** $f_Y(n) = Af_Y(n-1)$ *and* $f_Y(n) = A^n f_Y(0)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \\ Y_{n-1} & Y_{n-2} & Y_{n-3} \end{pmatrix}$$

*and*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{pmatrix}.$$

**(c)** $f_Y(n+m) = A^n f_Y(m)$ *and* $f_Y(n+m) = A^m f_Y(n)$ *i.e.,*

$$\begin{pmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{pmatrix},$$

*and*

$$\begin{pmatrix} Y_{m+n+2} & Y_{m+n+1} & Y_{m+n} \\ Y_{m+n+1} & Y_{m+n} & Y_{m+n-1} \\ Y_{m+n} & Y_{m+n-1} & Y_{m+n-2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix},$$

*and* $f_Y(n) = A^m f_Y(n-m)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{pmatrix}.$$

*Proof.* Set $W_n = Y_n$, and $r = 0$, $s = -1$, $t = 1$ in [8, Lemma 56]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

**Theorem 41.** *For all integers $n$ and $m$, the following identities hold.*

**(a)** *Catalan's Identity:*

$$\det(f_Y(n+m)) = \det(f_Y(m)) \quad and \quad \det(f_Y(n)) = \det(f_Y(n-m)),$$

*i.e.,*

$$\begin{vmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{vmatrix} = \begin{vmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{vmatrix},$$

*and*

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{vmatrix}.$$

**(b)** *(see Theorem 18) Simson's (or Cassini's) Identity:*

$$\det(f_Y(n)) = \det(f_Y(0)),$$

*i.e.,*

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}.$$

*Proof.* Set $W_n = Y_n$, and $r = 0$, $s = -1$, $t = 1$ in [8, Theorem 57]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of co-Narayana numbers (take $Y_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 0$).

**Corollary 42.** *For all integers $n$ and $m$, the following identities hold.*

**(a)** *Catalan's Identity:*

$$\det(f_M(n+m)) = \det(f_M(m)) \quad \textit{and} \quad \det(f_M(n)) = \det(f_M(n-m)),$$

*i.e.,*

$$\begin{vmatrix} M_{n+m+2} & M_{n+m+1} & M_{n+m} \\ M_{n+m+1} & M_{n+m} & M_{n+m-1} \\ M_{n+m} & M_{n+m-1} & M_{n+m-2} \end{vmatrix} = \begin{vmatrix} M_{m+2} & M_{m+1} & M_m \\ M_{m+1} & M_m & M_{m-1} \\ M_m & M_{m-1} & M_{m-2} \end{vmatrix},$$

*and*

$$\begin{vmatrix} M_{n+2} & M_{n+1} & M_n \\ M_{n+1} & M_n & M_{n-1} \\ M_n & M_{n-1} & M_{n-2} \end{vmatrix} = \begin{vmatrix} M_{n-m+2} & M_{n-m+1} & M_{n-m} \\ M_{n-m+1} & M_{n-m} & M_{n-m-1} \\ M_{n-m} & M_{n-m-1} & M_{n-m-2} \end{vmatrix}.$$

**(b)** *Simson's (or Cassini's) Identity:*

$$\det(f_M(n)) = \det(f_M(0)),$$

*i.e.,*

$$\begin{vmatrix} M_{n+2} & M_{n+1} & M_n \\ M_{n+1} & M_n & M_{n-1} \\ M_n & M_{n-1} & M_{n-2} \end{vmatrix} = -1.$$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -2$ in the last Theorem, we have the following Corollary which gives determinantal formulas of co-Narayana-Lucas numbers.

**Corollary 43.** *For all integers $n$ and $m$, the following identities hold.*

**(a)** *Catalan's Identity:*

$$\det(f_S(n+m)) = \det(f_S(m)) \quad \textit{and} \quad \det(f_S(n)) = \det(f_S(n-m))$$

*i.e.,*

$$\begin{vmatrix} S_{n+m+2} & S_{n+m+1} & S_{n+m} \\ S_{n+m+1} & S_{n+m} & S_{n+m-1} \\ S_{n+m} & S_{n+m-1} & S_{n+m-2} \end{vmatrix} = \begin{vmatrix} S_{m+2} & S_{m+1} & S_{m} \\ S_{m+1} & S_{m} & S_{m-1} \\ S_{m} & S_{m-1} & S_{m-2} \end{vmatrix},$$

*and*

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_{n} \\ S_{n+1} & S_{n} & S_{n-1} \\ S_{n} & S_{n-1} & S_{n-2} \end{vmatrix} = \begin{vmatrix} S_{n-m+2} & S_{n-m+1} & S_{n-m} \\ S_{n-m+1} & S_{n-m} & S_{n-m-1} \\ S_{n-m} & S_{n-m-1} & S_{n-m-2} \end{vmatrix}.$$

**(b)** *Simson's (or Cassini's) Identity:*

$$\det(f_S(n)) = \det(f_S(0)),$$

*i.e.,*

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_{n} \\ S_{n+1} & S_{n} & S_{n-1} \\ S_{n} & S_{n-1} & S_{n-2} \end{vmatrix} = -31.$$

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