Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 4, 2025, Pages 473-487
https://doi.org/10.34198/ejms.15425.473487

Power Series and Finite Element Methods for Solving Cahn-Hilliard Equation

Peter Oluwafemi Olatunji[1,*] and Richard Olu Awonusika[2]

[1] Department of Mathematical Sciences, Adekunle Ajasin University, P.M.B. 001, Akungba-Akoko, Ondo State, Nigeria
e-mail: peter.olatunji@aaua.edu.ng

[2] Department of Mathematical Sciences, Adekunle Ajasin University, P.M.B. 001, Akungba-Akoko, Ondo State, Nigeria
e-mail: richard.awonusika@aaua.edu.ng

Abstract

The Cahn-Hilliard equation is a nonlinear partial differential equation that describes spinodal decomposition, coarsening phenomena, and the dynamics of phase separation for ternary iron alloys. This article employs a power series technique and the finite element method to obtain analytical and numerical solutions of the Cahn-Hilliard equation, respectively. For the power series method, the nonlinear terms in the proposed problem are dealt with using the generalised Cauchy product of power series, which allows us to obtain an explicit recursion formula for the expansion function coefficient of the series solution. On the other hand, numerical solution to the Cahn-Hilliard equation is obtained using the finite element method that is based on the implicit time-stepping scheme and the sparse linear algebra technique. The obtained analytical and numerical solutions are compared with the exact solution to illustrate the accuracy and reliability of the proposed methods. The absolute errors obtained show that the proposed methods are accurate and reliable. Two and three dimensional graphs of the exact and approximate solutions are presented for comparison purposes.

1 Introduction

Nonlinear partial differential equations (PDEs) are indispensable tools in describing several real-life phenomena in applied sciences such as fluid mechanics, solid mechanics, engineering, hydrodynamics, electromagnetic theory, quantum mechanics, elasticity, and reaction-diffusion processes ([42]). In handling these nonlinear problems, advanced methods of solutions are required ([4], [35], [42], [51]). A number of authors have applied different advanced analytical and numerical methods to solve nonlinear PDEs arising in real-life situations. For instance, Laplace transform and Adomian decomposition method were used in [2], simple equation method was employed in [17] and [37], variation of parameters and

Received: February 28, 2025; Accepted: April 1, 2025; Published: April 7, 2025

2020 Mathematics Subject Classification: 33C45, 34A08, 34B16, 35R11, 65L05.

Keywords and phrases: Cahn-Hilliard equation, power series, generalised Cauchy product, nonlinear partial differential equation, finite element method, time-stepping scheme, sparse linear algebra technique.

*Corresponding author

characteristics methods were considered in [34], the first integral method was used in [26], exponential finite difference method was considered in [22], double Laplace-Sumudu transform coupled with iterative method was applied in [1]. For several others methods, see [27], [36], [43], [45], [46]. Linear multistep methods ([25], [38], [40], [41]) have also been applied to obtain numerical solutions to different classes of differential equations.

The Cahn-Hilliard equation is a nonlinear fourth-order partial differential equation that describes and models complicated phase separation, coarsening phenomena in a melted alloy, spinodal decomposition, and the dynamics of phase separation for ternary iron alloys. This equation, in its simplest, one dimensional form, is given by ([13], [44])

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2} - u^3(x,t) + u(x,t)\right) = 0, \quad x \in \mathbb{R}. \tag{1.1}$$

The Cahn-Hilliard equation (1.1) has further been used in the modelling of several other physical system phenomena, such as phase transitions in material science, polymer, and protein dynamics, and pattern formation in fluids. A special case of the Cahn-Hilliard equation is the famous Cahn-Allen equation ([6], [20], [23], [28])

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^3(x,t) - u(x,t) = 0, \tag{1.2}$$

which describes chemical reaction, Faraday instability, Rayleigh-Benard convection, and the interaction of different phases of matter, such as solids and liquids with respect to time t. Recently, Awonusika ([6]) applied a power series method based on the generalised Cauchy product to obtain approximate analytical solution of the Cahn-Allen equation (1.2) and its generalisation. For physical, mathematical, and numerical derivations of the Cahn-Hilliard equation, see [11], [12], [13], [14], [15], [16], [31], and the references therein.

Many authors have, in the past decades, applied several methods to obtain both analytical and numerical solutions of the Cahn-Hilliard equation. Alikakos et al. in [3] showed, using asymptotic expansion and spectral methods, that the level surfaces of solutions to the Cahn-Hilliard equation converges to the Hele-Shaw equation provided that the classical solutions of the latter exist. The author in [49] presented a detailed review on the well-posedness and long-time behaviour of global solutions of the Cahn-Hilliard equation. Hussain et al. [24] recently considered exact solutions of the Cahn-Hilliard equation in terms of Weierstrass-elliptic and Jacobi-elliptic functions using the F-expansion method. In [18], the authors studied the regularisation and strict separation properties of the unique solution of the Cahn-Hilliard-Oono equation. In [19], the authors considered energy law preserving method for the numerical investigation of the Cahn-Allen and Cahn-Hilliard equations. In [30], Kim et al. presented a review on the applications and computational simulation results of the Cahn-Hilliard equation. In [44], the author studied spinodal decomposition and coarsening properties of the Cahn-Hilliard equation. For other analytical and numerical investigations of the Cahn-Hilliard equation, see [29], [33], [50].

One of the challenges posed in solving PDEs numerically arise from its higher-order spatial derivatives and its nonlinear terms. Standard numerical methods such as finite differences can be inefficient or unstable without careful discretizations, making finite element method (FEM) an attractive alternative. The primary advantage of using FEM is its ability to handle complex geometries and boundary conditions more naturally than traditional finite difference methods. In this paper, we consider the analytical and numerical solutions of the Cahn-Hilliard equation ($0 \le x \le 1, 0 < t \le 1$) [48]

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2} - u^3(x,t) + u(x,t)\right) - \beta \frac{\partial u(x,t)}{\partial x} = 0, \quad \beta \in \mathbb{R}, \tag{1.3}$$

satisfying the initial condition

$$u(x,0) = u_0(x). \tag{1.4}$$

Ugurlu and Kaya in [48], used a modified extended tanh function method to obtain several exact solutions of the Cahn-Hilliard equation (1.3), and presented approximate solutions using a homotopy perturbation method together with the Adomian decomposition method. In this paper, we use a power series technique that is based on the generalised Cauchy product in respect of the nonlinear terms ([5], [6], [7], [8], [9]) and the finite element method ([21], [32], [47]) based on the implicit time-stepping scheme and the sparse linear algebra technique. If $\beta = 0$, then equation (1.3) reduces to the classical Cahn-Hilliard equation (1.1). The proposed power series approach does not require any type of polynomial or linearisation technique in the simplification of the nonlinear terms, as the generalised Cauch product will conveniently transform the higher power of series solution into another power series. Thus, an explicit recursion formula is obtained for the expansion coefficients of the series solution. These coefficients are space-variable expansion coefficients. A special case in which $\beta = 1$ is considered to illustrate the effectiveness, accuracy, and reliability of the proposed method. Our approximate solutions obtained from the proposed methods are compared with the exact solution. The absolute errors obtained show that the proposed methods are effective, accurate, and reliable. Two and three dimensional graphs of the exact and approximate solutions are presented to illustrate the proposed methods' reliability and accuracy.

2 Power Series Method of Solution

In this section, we present the proposed power series method of obtaining the analytical solution of the Cahn-Hilliard equation (1.3) satisfying the initial condition (1.4). In the proposed method, one assumes that the solution $u(x,t)$ assumes a power series in t with x-variable expansion coefficients. Upon substituting the assumed series solution into the governing equation requires one to apply the generalised Cauchy product of multiple power series. Interestingly, the Cauchy product of these series is again a power series.

Theorem 2.1. *For $\beta \in \mathbb{R}$, the Cahn-Hilliard initial value problem* $(0 \leq x \leq 1, 0 < t \leq 1)$

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2} - u^3(x,t) + u(x,t)\right) - \beta\frac{\partial u(x,t)}{\partial x} = 0, \quad \beta \in \mathbb{R}, \\ &u(x,0) = u_0(x), \end{aligned} \tag{2.1}$$

admits the series solution

$$u(x,t) = \sum_{\ell=0}^{\infty} a_\ell(x)t^\ell = u_0(x) + a_1(x)t + a_2(x)t^2 + a_3(x)t^3 + \cdots; \tag{2.2}$$

with the expansion function coefficients $a_\ell(x)$ $(\ell = 1, 2, \dots)$ *given recursively by*

$$a_{(\ell+1)}(x) = \frac{\mathsf{A}_{\ell,3}(x) + 6\mathsf{B}_{\ell,3}(x) - a_\ell^{(4)}(x) - a_\ell''(x) + \beta a_\ell'(x)}{\ell+1}, \ \ell = 0, 1, 2, \dots. \tag{2.3}$$

Here the variable coefficients $\mathsf{A}_{\ell,3}(x)$ *and* $\mathsf{B}_{\ell,3}(x)$ $(\ell = 0, 1, 2, \dots)$ *are given, respectively, by*

$$\mathsf{A}_{\ell,3}(x) = \sum_{p=0}^{\ell}\sum_{q=0}^{p} a_q(x)a_{p-q}(x)a''_{\ell-p}(x), \qquad \mathsf{A}_{0,3}(x) = (u_0(x))^2 u_0'', \tag{2.4}$$

$$\mathsf{B}_{\ell,3}(x) = \sum_{p=0}^{\ell}\sum_{q=0}^{p} a'_q(x)a'_{p-q}(x)a_{\ell-p}(x), \qquad \mathsf{B}_{0,3}(x) = (u_0'(x))^2 u_0. \tag{2.5}$$

Proof. Differentiating, one has

$$\left(-u_{xx} + u^3 - u\right)_{xx} = -u_{xxxx} + 3u^2 u_{xx} + 6\,(u_x)^2\, u - u_{xx}. \tag{2.6}$$

Assuming a formal power series solution in t (about $t = 0$):

$$u(x,t) = \sum_{\ell=0}^{\infty} a_\ell(x)t^\ell = a_0(x) + a_1(x)t + a_2(x)t^2 + a_3(x)t^3 + \cdots, \tag{2.7}$$

one has the following differentiation formulae.

$$\frac{\partial u(x,t)}{\partial t} = \sum_{\ell=1}^{\infty} \ell a_\ell(x)t^{\ell-1} = \sum_{\ell=0}^{\infty}(\ell+1)a_{\ell+1}(x)t^\ell \tag{2.8}$$

$$\frac{\partial u(x,t)}{\partial x} = \sum_{\ell=0}^{\infty} a'_\ell(x)t^\ell = a'_0(x) + a'_1(x)t + a'_2(x)t^2 + a'_3(x)t^3 + \cdots \tag{2.9}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{\ell=0}^{\infty} a''_\ell(x)t^\ell = a''_0(x) + a''_1(x)t + a''_2(x)t^2 + a''_3(x)t^3 + \cdots \tag{2.10}$$

$$\frac{\partial^4 u(x,t)}{\partial x^4} = \sum_{\ell=0}^{\infty} a^{(4)}_\ell(x)t^\ell = a^{(4)}_0(x) + a^{(4)}_1(x)t + a^{(4)}_2(x)t^2 + a^{(4)}_3(x)t^3 + \cdots. \tag{2.11}$$

The initial value problem (2.1) now becomes

$$\begin{aligned} &u_t = -u_{xxxx} + 3u^2 u_{xx} + 6\,(u_x)^2\, u - u_{xx} + \beta u_x, \quad 0 < x \le 1, \\ &u(x,0) = u_0(x). \end{aligned} \tag{2.12}$$

Upon substituting the differentiation formulae (2.8)-(2.11) into the governing equation (2.12), we get

$$\begin{aligned} &\sum_{\ell=0}^{\infty}(\ell+1)a_{\ell+1}(x)t^{\ell} = -\sum_{\ell=0}^{\infty} a_{\ell}^{(4)}(x)t^{\ell} + 3\left(\sum_{\ell=0}^{\infty} a_{\ell}(x)t^{\ell}\right)^2 \left(\sum_{\ell=0}^{\infty} a_{\ell}''(x)t^{\ell}\right) \\ &+6\left(\sum_{\ell=0}^{\infty} a_{\ell}'(x)t^{\ell}\right)^2 \left(\sum_{\ell=0}^{\infty} a_{\ell}(x)t^{\ell}\right) - \sum_{\ell=0}^{\infty} a_{\ell}''(x)t^{\ell} + \beta\sum_{\ell=0}^{\infty} a_{\ell}'(x)t^{\ell}. \end{aligned} \tag{2.13}$$

Using the Cauchy product ([5]), we have

$$\begin{aligned} \left(\sum_{\ell=0}^{\infty} a_{\ell}(x)t^{\ell}\right)^2 \left(\sum_{\ell=0}^{\infty} a_{\ell}''(x)t^{\ell}\right) &= \sum_{\ell=0}^{\infty} \mathsf{A}_{\ell,3}(x)t^{\ell} \\ \left(\sum_{\ell=0}^{\infty} a_{\ell}'(x)t^{\ell}\right)^2 \left(\sum_{\ell=0}^{\infty} a_{\ell}(x)t^{\ell}\right) &= \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,3}(x)t^{\ell}, \end{aligned}$$

where the coefficients $\mathsf{A}_{\ell,3}(x)$ and $\mathsf{B}_{\ell,3}(x)$ are given, respectively, by

$$\mathsf{A}_{\ell,3}(x) = \sum_{p=0}^{\ell}\sum_{q=0}^{p} a_q(x)a_{p-q}(x)a_{\ell-p}''(x), \quad \mathsf{B}_{\ell,3}(x) = \sum_{p=0}^{\ell}\sum_{q=0}^{p} a_q'(x)a_{p-q}'(x)a_{\ell-p}(x). \tag{2.14}$$

Thus equation (2.13) becomes

$$\begin{aligned} &\sum_{\ell=0}^{\infty}(\ell+1)a_{\ell+1}(x)t^{\ell} = -\sum_{\ell=0}^{\infty} a_{\ell}^{(4)}(x)t^{\ell} + 3\sum_{\ell=0}^{\infty} \mathsf{A}_{\ell,3}(x)t^{\ell} + 6\sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,3}(x)t^{\ell} \\ &-\sum_{\ell=0}^{\infty} a_{\ell}''(x)t^{\ell} + \beta\sum_{\ell=0}^{\infty} a_{\ell}'(x)t^{\ell}. \end{aligned} \tag{2.15}$$

Equating the coefficients of t^{ℓ} $(\ell = 0, 1, 2, \dots)$ in (2.15), we obtain

$$(\ell+1)a_{\ell+1}(x) = -a_{\ell}^{(4)}(x) + 3\mathsf{A}_{\ell,3}(x) + 6\mathsf{B}_{\ell,3}(x) - a_{\ell}''(x) + \beta a_{\ell}'(x). \tag{2.16}$$

Hence, we obtain the recurrence relation for $a_{\ell+1}(x)$ $(\ell = 0, 1, 2, \dots)$ as required:

$$a_{\ell+1}(x) = \frac{3\mathsf{A}_{\ell,3}(x) + 6\mathsf{B}_{\ell,3}(x) - a_{\ell}^{(4)}(x) - a_{\ell}''(x) + \beta a_{\ell}'(x)}{\ell+1}. \tag{2.17}$$

□

Corollary 2.2. *The coefficients $a_\ell(x)$ ($\ell = 1, 2, \dots$) given by the recurrence relation (2.17) admit the following first values:*

$$\begin{aligned} a_1(x) =& 3\mathsf{A}_{0,3}(x) + 6\mathsf{B}_{0,3}(x) - a_0^{(4)}(x) - a_0''(x) + \beta a_0'(x) \\ =& \beta u_0'(x) + 6u_0(x)(u_0'(x))^2 + 3(u_0(x))^2 u_0''(x) - u_0''(x) - {u_0}^{(4)}(x) \end{aligned} \tag{2.18}$$

$$\begin{aligned} a_2(x) =& \frac{3\mathsf{A}_{1,3}(x) + 6\mathsf{B}_{1,3}(x) - a_1^{(4)}(x) - a_1''(x) + \beta a_1'(x)}{2} \\ =& \frac{\beta a_1'(x) + 12u_0(x)u_0'(x)a_1'(x) + 6a_1(x)(u_0'(x))^2 + 3(u_0(x))^2 a_1''(x)}{2} \\ &+ \frac{6a_1(x)u_0(x)u_0''(x) - a_1''(x) - {a_1}^{(4)}(x)}{2} \end{aligned} \tag{2.19}$$

$$\begin{aligned} a_3(x) =& \frac{\beta a_2'(x) + 6u_0(x)(a_1'(x))^2 + 12u_0(x)u_0'(x)a_2'(x) + 6a_2(x)(u_0'(x))^2}{3} \\ &+ \frac{12a_1(x)u_0'(x)a_1'(x) + 3(u_0(x))^2 a_2''(x) + 6a_2(x)u_0(x)u_0''(x)}{3} \\ &+ \frac{6a_1(x)u_0(x)a_1''(x) + 3(a_1(x))^2 u_0''(x) - a_2''(x) - {a_2}^{(4)}(x)}{3}. \end{aligned} \tag{2.20}$$

3 Finite Element Method of Solution

In this section, we present numerical solutions of the Cahn-Hilliard problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2} - u^3(x,t) + u(x,t)\right) - \beta\frac{\partial u(x,t)}{\partial x} = 0, \quad \beta \in \mathbb{R}, \\ &u(x,0) = u_0(x) \end{aligned} \tag{3.1}$$

using the finite element method (FEM) based on discretization. To this end, multiplying the problem (3.1) by a test function v and integrating over the domain x, one obtains the weak formulation

$$\int_0^1 \left(\frac{\partial u}{\partial t}v + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2} - u^3 + u\right)v - \beta\frac{\partial u}{\partial x}v\right)dx = 0. \tag{3.2}$$

We discretize the spatial domain using piecewise linear basis functions. For the time integration, we employ an implicit time-stepping scheme, such as the backward Euler method, which provides improved stability for stiff problems (see [10], [39]) such as the Cahn-Hilliard equation. The discrete form of the equation can be written as

$$M\frac{u^{n+1} - u^n}{\Delta t} + Au^{n+1} - A(u^{n+1})^3 + Au^{n+1} - B\beta u^{n+1} = 0, \tag{3.3}$$

where M is the mass matrix arising from the time derivative term, A represents the discrete Laplacian operator, and B denotes the first derivative term, defined, respectively, as

$$M_{ij} = \int_0^1 \omega_i\omega_j dx, \qquad A_{ij} = \int_0^1 \left(\frac{d^2\omega_i}{dx^2}\frac{d^2\omega_j}{dx^2}\right)dx, \qquad B_{ij} = \int_0^1 \frac{d\omega_i}{dx}\omega_j dx. \tag{3.4}$$

where ω_i and ω_j are the basis functions. In the implementation, the nonlinear system obtained is solved using Newtonâs method.

4 Example

This section presents an illustrative special case of our main result in Theorem 2.1 as well as the finite element method algorithm presented in Section 3. We give explicit series solutions and numerical solutions of the initial value problem (1.3) with $\beta = 1$. Higher expansion coefficients of the series solution of the proposed example are obtained using Wolfram Mathematica software 12.0.

The absolute error, $\mathscr{AE}$, is defined by

$$\mathscr{AE} = \left| u_{\text{ex.}}(x,t) - u_{\text{appr.}}(x,t) \right|, \quad 0 < x \leq 1, t > 0. \tag{4.1}$$

Here, $u_{\text{ex.}}(x,t)$ represents the exact solution and $u_{\text{appr.}}(x,t)$ denotes the approximate solution.

Now, consider the Cahn-Hilliard initial value problem ([48])

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2} - u^3(x,t) + u(x,t)\right) - \frac{\partial u(x,t)}{\partial x} = 0, \quad 0 < x \leq 1, \\ &u(x,0) = \tanh\left(\frac{\sqrt{2}}{2}x\right). \end{aligned} \tag{4.2}$$

The exact solution of the problem (4.2) is

$$u(x,t) = \tanh\left[\frac{\sqrt{2}}{2}(x+t)\right]. \tag{4.3}$$

PSM. Upon setting $\beta = 1$, $u_0(x) = \phi = \tanh((\sqrt{2}/2)x)$, $\psi = \operatorname{sech}((\sqrt{2}/2)x)$ in Corollary 2.2, we obtain

$$\begin{aligned} u(x,t) =& \phi + \frac{1}{2}\psi^2\left(\sqrt{2} - 2\phi\left(\psi^2 + \phi^2 - 1\right)\right)t \\ &+ \left[-\frac{\psi^6}{\sqrt{2}} + \frac{\psi^4}{\sqrt{2}} - \psi^2\phi^7 - 12\psi^4\phi^5 + 2\psi^2\phi^5 + \sqrt{2}\psi^2\phi^4 + \frac{3\psi^6\phi^3}{2} + 10\psi^4\phi^3\right]t^2 \\ &+ \left[-\psi^2\phi^3 + \frac{\psi^4\phi^2}{\sqrt{2}} - \sqrt{2}\psi^2\phi^2 + \frac{25\psi^8\phi}{2} - \frac{29\psi^6\phi}{2} + 2\psi^4\phi - \frac{\psi^2\phi}{2}\right]t^2 + \cdots. \end{aligned} \tag{4.4}$$

For numerical comparison purposes, we compute the approximate series solution

$$\begin{aligned} u_{\mathrm{PSM}}(x,t) =& \phi + \frac{1}{2}\psi^2\left(\sqrt{2} - 2\phi\left(\psi^2 + \phi^2 - 1\right)\right)t \\ &+ \left[-\frac{\psi^6}{\sqrt{2}} + \frac{\psi^4}{\sqrt{2}} - \psi^2\phi^7 - 12\psi^4\phi^5 + 2\psi^2\phi^5 + \sqrt{2}\psi^2\phi^4 + \frac{3\psi^6\phi^3}{2} + 10\psi^4\phi^3\right]t^2 \\ &+ \left[-\psi^2\phi^3 + \frac{\psi^4\phi^2}{\sqrt{2}} - \sqrt{2}\psi^2\phi^2 + \frac{25\psi^8\phi}{2} - \frac{29\psi^6\phi}{2} + 2\psi^4\phi - \frac{\psi^2\phi}{2}\right]t^2 \\ &+ a_3t^3 + \cdots + a_{10}t^{10} \end{aligned} \tag{4.5}$$

using Wolfram Mathematica. The numerical and graphical comparisons of the exact solution (4.3) and the approximate solution (4.5) are presented in Table 1 and Figures 1-2.

FEM. The FEM is also implemented in Python using sparse matrix techniques for computational efficiency. The numerical results are compared with the exact solution at different time steps, as shown in Table 2. A 2D graph and a 3D surface plot of the FEM solution $u(x,t)$ are shown in Figure 3.

5 Results and Discussion

In this paper, we use a power series method and the finite element method to obtain analytical and numerical solutions of the Cahn-Hilliard initial value problem (1.3)-(1.4), respectively. In the case of the power series method, the nonlinear terms are handled using the generalised Cauchy product, which in turn, enables us to construct an explicit recursion formula for the expansion variable coefficient of the series solution. For the finite element method, the implementation leverages sparse matrix structures, which significantly reduce memory usage and improve computation speed. The method scales well with increased problem size, making it suitable for large-scale simulations.

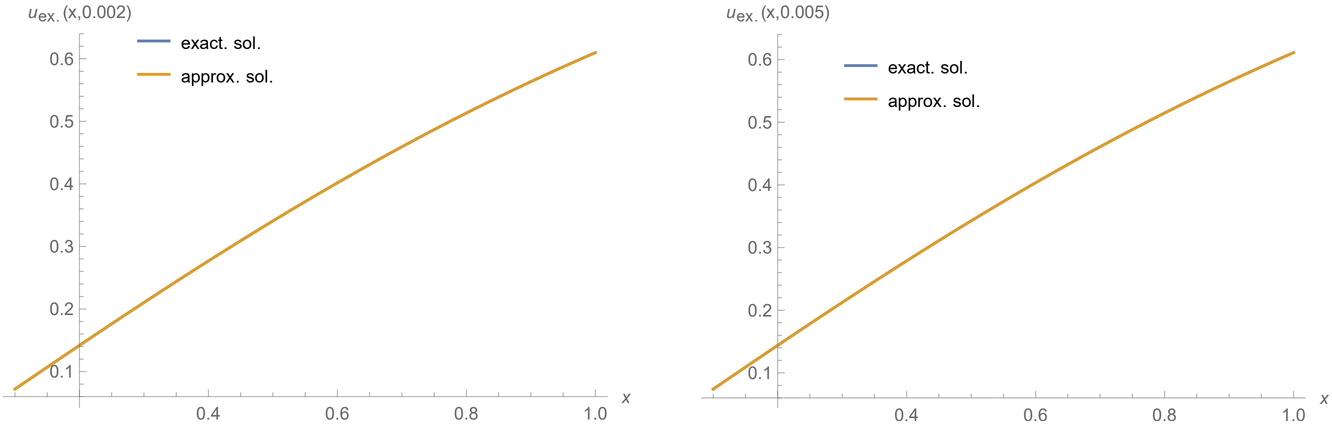
In the example presented, the approximate solutions obtained are compared with the given exact solution. Numerical illustrations of results are presented in Tables 1 and 2. Two dimensional graphs of power series solutions are illustrated in Figures 1a and 1b for $t = 0.002, t = 0.005$, respectively. Three dimensional graphs of exact and approximate series solutions are demonstrated in Figures 2a and 2b, respectively. While two and three dimensional graphs of finite element solutions are presented in Figures 3a and 3b, respectively. It is clearly observed from these tables and graphs that the exact and approximate solutions agree excellently, which is a clear indication that the present methods are efficient, accurate, and reliable in obtaining approximate solutions of nonlinear partial differential equations arising in real-life applications.

Table 1: Comparison of exact and approximate (PSM) solutions of Cahn-Hilliard equation (4.2) for $t = 0.002, t = 0.005$.

x	$u_{\text{ex.}}(x,t)$	$u_{\text{PSM}}(x,t)$	$\mathscr{AE}$	$u_{\text{ex.}}(x,t)$	$u_{\text{PSM}}(x,t)$	$\mathscr{AE}$
0.1	0.072000	0.072000	1.38×10^{-17}	0.074110	0.074110	0.000000
0.2	0.141872	0.141872	0.000000	0.143950	0.143950	3.08×10^{-15}
0.3	0.210358	0.210358	2.77×10^{-17}	0.212385	0.212385	3.58×10^{-13}
0.4	0.276840	0.276840	5.55×10^{-17}	0.278798	0.278798	4.95×10^{-13}
0.5	0.340774	0.340774	2.77×10^{-16}	0.342647	0.342647	2.06×10^{-12}
0.6	0.401703	0.401703	5.55×10^{-17}	0.403481	0.403481	4.13×10^{-13}
0.7	0.459273	0.459273	1.11×10^{-16}	0.460945	0.460945	8.43×10^{-13}
0.8	0.513226	0.513226	2.22×10^{-16}	0.514787	0.514787	7.10×10^{-13}
0.9	0.563407	0.563407	1.11×10^{-16}	0.564853	0.564853	1.19×10^{-12}
1.0	0.609749	0.609749	1.11×10^{-16}	0.611079	0.611079	7.10×10^{-13}

Table 2: Comparison of exact and approximate (FEM) solutions of Cahn-Hilliard equation (4.2) for $t = 0.002, t = 0.005$.

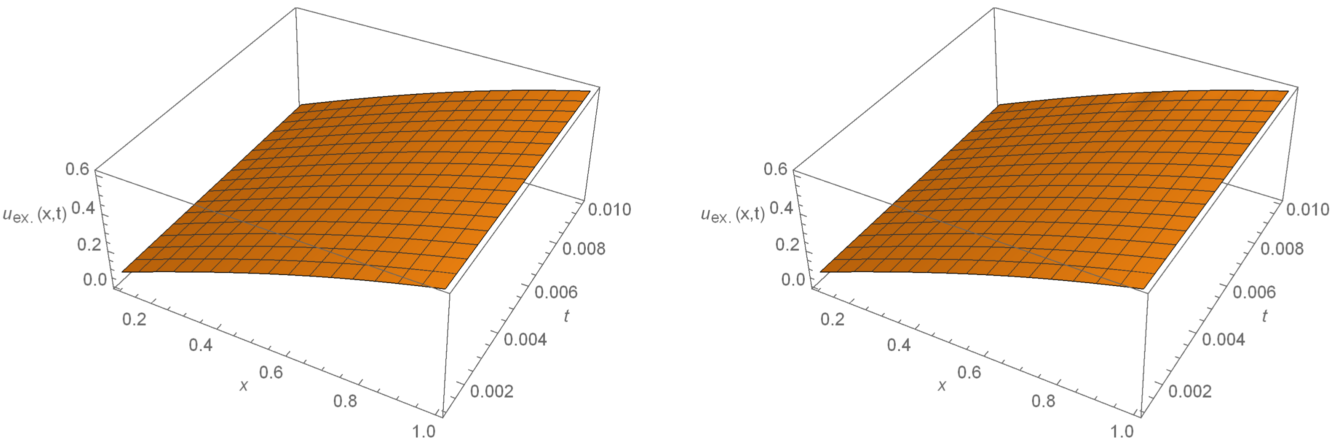
x	$u_{ex.}(x,t)$	$u_{\text{FEM}}(x,t)$	$\mathscr{AE}$	$u_{ex.}(x,t)$	$u_{\text{FEM}}(x,t)$	$\mathscr{AE}$
0.1	0.07200	0.070592	1.40×10^{-3}	0.074110	0.070593	6.8×10^{-2}
0.2	0.141872	0.140482	1.39×10^{-3}	0.143950	0.140486	3.5×10^{-3}
0.3	0.210358	0.208998	1.36×10^{-3}	0.212385	0.209006	3.4×10^{-3}
0.4	0.276840	0.275521	1.32×10^{-3}	0.278798	0.275534	3.2×10^{-3}
0.5	0.340774	0.339507	1.30×10^{-3}	0.342647	0.339523	3.1×10^{-3}
0.6	0.401703	0.400500	1.20×10^{-3}	0.403481	0.400516	3.0×10^{-3}
0.7	0.459273	0.458140	1.13×10^{-3}	0.460945	0.458156	2.8×10^{-3}
0.8	0.513226	0.512171	1.06×10^{-3}	0.514787	0.512183	2.6×10^{-3}
0.9	0.563407	0.562433	9.73×10^{-4}	0.564853	0.562441	2.4×10^{-3}
1.0	0.609749	0.608859	8.89×10^{-3}	0.611079	0.608859	2.2×10^{-3}



(a) Exact and power series solutions for $t = 0.002$.

(b) Exact and power series solutions for $t = 0.005$.

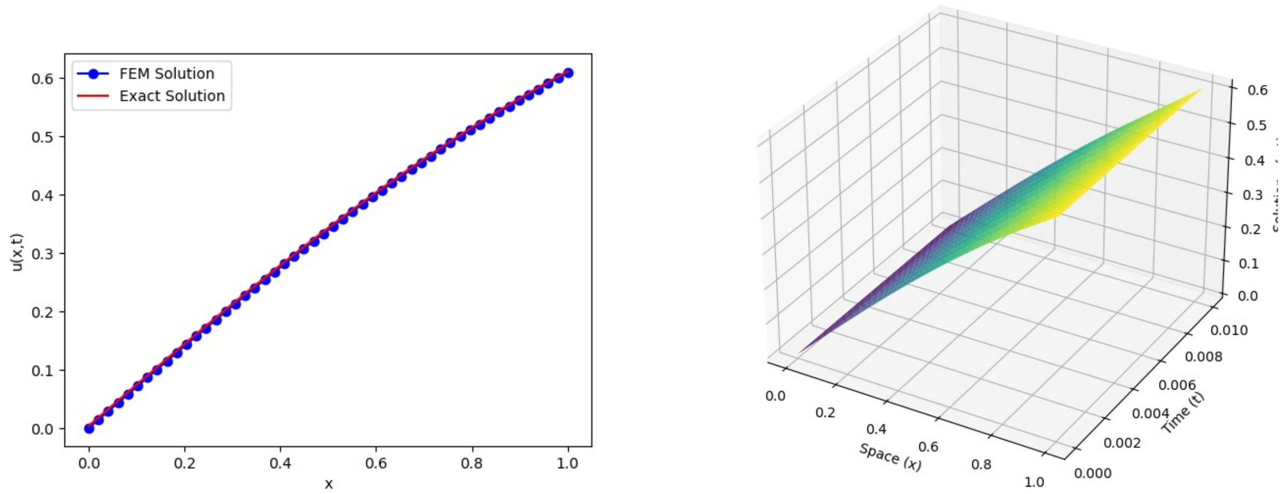
Figure 1: 2D graphs of exact and power series solutions of the Cahn-Hilliard equation (4.2).



(a) Exact solution.

(b) Approximate power series solution.

Figure 2: 3D graphs of exact and power series solutions of the Cahn-Hilliard equation (4.2).



(a) Exact and finite element solutions.

(b) 3D finite element solution.

Figure 3: 2D and 3D graphs of exact and approximate finite element solutions of the Cahn-Hilliard equation (4.2).

6 Concluding Remarks

This paper investigated the Cahn-Hilliard equation that describes pattern formation, comprehension of phase transitions, chemical reaction, and the interaction of different phases of matter. Using a power series technique, the series solution of the Cahn-Hilliard equation was obtained. A special case of our main result was considered to illustrate the effectiveness, reliability, and accuracy of the present power series method. The numerical solutions obtained using the FEM agree with the exact solution. However, the solutions do not have comparable accuracy to those using the PSM. This is not far fetched, because the present FEM converges with a quadratic rate of convergence with order 2, while the rate of convergence of the present PSM is of order 10.

References

[1] Ahmed, S. A., Qazza, A., & Saadeh, R. (2022). Exact solutions of nonlinear partial differential equations via the new double integral transform combined with iterative method. *Axioms, 11*(1), 247. https://doi.org/10.3390/axioms11060247

[2] Aland, P. V., & Singh, P. (2022). Solution of non-linear partial differential equations using Laplace transform modified Adomian decomposition method. *Journal of Physics: Conference Series, 2267*(1), 012156. https://doi.org/10.1088/1742-6596/2267/1/012156

[3] Alikakos, N. D., Bates, P. W., & Chen, X. (1994). Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Archive for Rational Mechanics and Analysis, 128*(2), 165-205. https://doi.org/10.1007/BF00375025

[4] Arrigo, D. J. (2022). *Analytical techniques for solving nonlinear partial differential equations, Synthese lectures on mathematics and statistics*. Springer Nature Switzerland AG. https://doi.org/10.1007/978-3-031-17069-0

[5] Awonusika, R. O. (2022). Analytical solution of a class of fractional Lane-Emden equation: A power series method. *International Journal of Applied and Computational Mathematics, 8*, 155. https://doi.org/10.1007/s40819-022-01354-w

[6] Awonusika, R. O. (2025). Approximate analytical solution of a class of highly nonlinear time-fractional-order partial differential equations. *Partial Differential Equations in Applied Mathematics, 13*, 101090. https://doi.org/10.1016/j.padiff.2025.101090

[7] Awonusika, R. O., & Mogbojuri, O. A. (2022). Approximate analytical solution of fractional Lane-Emden equation by Mittag-Leffler function method. *Journal of Nigerian Society of Physical Sciences, 4*, 265-280. https://doi.org/10.46481/jnsps.2022.687

[8] Awonusika, R. O., & Onuoha, O. B. (2024). Analytical method for systems of nonlinear singular boundary value problems. *Partial Differential Equations in Applied Mathematics, 11*, 100762. https://doi.org/10.1016/j.padiff.2024.100762

[9] Awonusika, R. O., & Onuoha, O. B. (2024). Analytical solution of system of fractional order Lane-Emden type equations. *Communications in Nonlinear Analysis, 1*, 1-25.

[10] Awonusika, R. O., & Olatunji, P. O. (2022). Analytical and numerical solutions of a class of generalized Lane-Emden equations. *Journal of Korean Society for Industrial and Applied Mathematics, 26*, 185-223.

[11] Bates, P. W., & Fife, P. C. (1993). The dynamics of nucleation for the Cahn-Hilliard equation. *SIAM Journal on Applied Mathematics, 53*, 990-1008. https://doi.org/10.1137/0153049

[12] Cahn, J. W. (1961). On the spinodal decomposition. *Acta Metallurgica, 9*, 795-801. https://doi.org/10.1016/0001-6160(61)90182-1

[13] Cahn, J. W., & Hilliard, J. E. (1958). Free energy of a nonuniform system I: Interfacial free energy. *Journal of Chemical Physics, 28*, 258-267. https://doi.org/10.1063/1.1744102

[14] Dias, L. S. (2023). Continuous theory of phase separation: The Cahn-Hilliard equations. *Revista Brasileira de Ensino de Física, 45*, e20230280. https://doi.org/10.1590/1806-9126-RBEF-2023-0280

[15] Elliott, C. M., & French, D. A. (1987). Numerical studies of the Cahn-Hilliard equation for phase separation. *IMA Journal of Applied Mathematics, 38*, 97-128. https://doi.org/10.1093/imamat/38.2.97

[16] Fife, P. C. (1991). Dynamical aspects of the Cahn-Hilliard equations. *Barter Lectures, University of Tennessee, Spring, 1991.*

[17] González-Gaxiola, O. (2022). Solution of nonlinear partial differential equations by Adomian decomposition method. *Studies in Engineering and Exact Sciences, 3*, 1-8. https://doi.org/10.54021/seesv3n1-007

[18] Giorgini, A., Grasselli, M., & Miranville, A. (2017). The Cahn-Hilliard-Oono equation with singular potential. *Mathematical Models and Methods in Applied Sciences, 27*, 2485-2510. https://doi.org/10.1142/s0218202517500506

[19] Hua, J., Lin, P., Liu, C., & Wang, Q. (2011). Energy law preserving C^0 finite element schemes for phase field models in two-phase flow computations. *Journal of Computational Physics, 230*, 7115-7131. https://doi.org/10.1016/j.jcp.2011.05.013

[20] Hariharan, G. (2014). An efficient Legendre wavelet-based approximation for a few Newell-Whitehead and Allen-Cahn equations. *Journal of Membrane Biology, 247*, 371-380. https://doi.org/10.1007/s00232-014-9638-z

[21] Hauser, J. R. (2009). *Numerical methods for nonlinear engineering models*. Springer, pp. 883-987. https://doi.org/10.1007/978-1-4020-9920-5

[22] Hilal, N., & Injrou, R., & Karroum, R. (2020). Exponential finite difference methods for solving Newell-Whitehead-Segel equation. *Arabian Journal of Mathematics*. https://doi.org/10.1007/s40065-020-00280-3

[23] Hussain, S., Shah, S. A., Ayub, S., & Ullah, A. (2019). An approximate analytical solution of the Allen-Cahn equation using homotopy perturbation method and homotopy analysis method. *Heliyon, 5*, e03060. https://doi.org/10.1016/j.heliyon.2019.e03060

[24] Hussain, A., Ibrahim, T. F., Birkea, F. M. O., Alotaibi, A. M., Al-Sinan, B. R., & Mukalazi, H. (2024). Exact solutions for the Cahn-Hilliard equation in terms of Weierstrass-elliptic and Jacobi-elliptic functions. *Scientific Reports, 14*, 13100. https://doi.org/10.1038/s41598-024-62961-9

[25] Igbinovia, E., Ogunfeyitimi, S. E., & Ikhile, M. N. O. (2025). Block hybrid trapezoidal-type methods for solving initial value problems in ordinary differential equations. *Earthline Journal of Mathematical Sciences, 15*, 345-365. https://doi.org/10.34198/ejms.15325.345365

[26] Jafari, H., Soltani, R., Khalique, C. M., & Baleanu, D. (2013). Exact solutions of two nonlinear partial differential equations by using the first integral method. *Boundary Value Problems, 2013*, 117. https://doi.org/10.1186/1687-2770-2013-117

[27] Khalid, M., & Khan, F. S. (2017). A new approach for solving highly nonlinear partial differential equations by successive differentiation method. *Mathematical Methods in the Applied Sciences*, 1-8. https://doi.org/10.1002/mma.4421

[28] Khan, I., Nawaz, R., Ali, A. H., Akgul, A., & Lone, S. A. (2023). Comparative analysis of the fractional order Cahn-Allen equation. *Partial Differential Equations in Applied Mathematics, 8*, 100576. https://doi.org/10.1016/j.padiff.2023.100576

[29] Kim, J. (2007). A numerical method for the Cahn-Hilliard equation with a variable mobility. *Communications in Nonlinear Science and Numerical Simulation, 12*, 1560. https://doi.org/10.1016/j.cnsns.2006.02.010

[30] Kim, J., Lee, S., Choi, Y., Lee, S., & Jeong, D. (2016). Basic principles and practical applications of the Cahn-Hilliard equation. *Mathematical Problems in Engineering, 2016*, 1-11. https://doi.org/10.1155/2016/9532608

[31] Lee, D., Huh, J.-Y., Jeong, D., Shin, J., Yun, A., & Kim, J. (2014). Physical, mathematical, and numerical derivations of the Cahn-Hilliard equation. *Computational Materials Science, 81*, 216-225. https://doi.org/10.1016/j.commatsci.2013.08.027

[32] Larson, M. G., & Bengzon, F. (2013). *The finite element method: Theory, implementation, and applications*. Springer. https://doi.org/10.1007/978-3-642-33287-6

[33] de Mello, E. V. L., Otton, T., & da Silveira, F. (2005). Numerical study of the Cahn-Hilliard equation in one, two, and three dimensions. *Physica A, 347*, 429. https://doi.org/10.1016/j.physa.2004.08.076

[34] Mhadhbi, N., Gana, S., & Alsaeedi, M. F. (2024). Exact solutions for nonlinear partial differential equations via a fusion of classical methods and innovative approaches. *Scientific Reports, 14*, 6443. https://doi.org/10.1038/s41598-024-57005-1

[35] Nair, S. (2011). *Advanced topics in applied mathematics: For engineering and the physical sciences*. Cambridge University Press. https://doi.org/10.1017/CBO9780511976995

[36] Nirmala, A. N., & Kumbinarasaiah, S. (2024). Numerical solution of nonlinear Hunter-Saxton equation, Benjamin-Bona Mahony equation, and Klein-Gordon equation using Hosoya polynomial method. *Results in Control and Optimization, 14*, 100388. https://doi.org/10.1016/j.rico.2024.100388

[37] Nofal, T. A. (2016). Simple equation method for nonlinear partial differential equations and its applications. *Journal of the Egyptian Mathematical Society, 24*, 204-209. https://doi.org/10.1016/j.joems.2015.05.006

[38] Olatunji, P. O., & Ikhile, M. N. O. (2020). Variable order nested hybrid multistep methods for stiff ODEs. *Journal of Mathematical and Computational Science, 10*(1), 78-94. https://doi.org/10.28919/jmcs/4147

[39] Olatunji, P. O., & Ikhile, M. N. O. (2020). Strongly regular general linear methods. *Journal of Scientific Computing, 82*(7), 1-25. https://doi.org/10.1007/s10915-019-01107-w

[40] Olatunji, P. O., & Ikhile, M. N. O. (2021). FSAL mono-implicit Nordsieck general linear methods with inherent Runge-Kutta stability for DAEs. *Journal of the Korean Society for Industrial and Applied Mathematics, 25*(4), 262-295.

[41] Olatunji, P. O., Ikhile, M. N. O., & Okuonghae, R. I. (2021). Nested second derivative two-step Runge-Kutta methods. *International Journal of Applied and Computational Mathematics, 7*, 1-39. https://doi.org/10.1007/s40819-021-01169-1

[42] Roubíček, T. (2005). *Nonlinear partial differential equations with applications*. Birkhäuser Verlag.

[43] Salas, A. H. (2012). Solving nonlinear partial differential equations by the sn-ns method. *Abstract and Applied Analysis, 2012*, 340824, 25 pages. https://doi.org/10.1155/2012/340824

[44] Scheel, A. (2015). Spinodal decomposition and coarsening fronts in the Cahn-Hilliard equation. *Journal of Dynamics and Differential Equations*. https://doi.org/10.1007/s10884-015-9491-5

[45] Singh, I., & Kumar, S. (2017). Efficient hybrid method for solving special type of nonlinear partial differential equations. *Numerical Methods for Partial Differential Equations*, 29 pages. https://doi.org/10.1002/num.22227

[46] Singh, P., & Sharma, D. (2018). Convergence and error analysis of series solution of nonlinear partial differential equation. *Nonlinear Engineering, 7*, 303-308. https://doi.org/10.1515/nleng-2017-0113

[47] Tekkaya, A. E., & Soyarslan, C. (2014). Finite element method. In L. Laperrière & G. Reinhart (Eds.), *CIRP Encyclopedia of Production Engineering*. Springer. https://doi.org/10.1007/978-3-642-20617-7_16699

[48] Ugurlu, Y., & Kaya, D. (2008). Solutions of the Cahn-Hilliard equation. *Computers and Mathematics with Applications, 56*, 3038-3045. https://doi.org/10.1016/j.camwa.2008.07.007

[49] Wu, H. (2022). A review on the Cahn-Hilliard equation: Classical results and recent advances in dynamic boundary conditions. *Electronic Research Archive, 30*, 2788-2832. https://doi.org/10.3934/era.2022143

[50] Li, D., & Zhong, C. (1998). Global attractor for the Cahn-Hilliard system with fast growing nonlinearity. *Journal of Differential Equations, 149*, 191. https://doi.org/10.1006/jdeq.1998.3429

[51] Yakushevich, L. V. (1998). *Nonlinear Physics of DNA*. Wiley, New York, NY, USA.