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Fekete-Szegö Problem for Univalent Functions and Quasiconformal Extension

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Abstract

Via Löwner theories, by Becker's and Betker's conditions on Herglotz function which give sufficient conditions for univalent functions admitting k-quasiconformal extension to the complex plane, we define two subclasses denoted by S_k^B and S_k^{BT}. Then we solve the Fekete-Szegö problem on these two subclasses.

1 Introduction

Given a real number $\lambda \in \mathbb{R}$, the Fekete-Szegö problem for analytic function class $\mathcal{F}$ is to find the maximum of $\max_{f\in\mathcal{F}} |a_3 - \lambda a_2^2|$, where a_2 and a_3 are Taylor coefficients of f. Fekete-Szegö problem plays an important role in characterizing the geometric quantities of functions f, for example, $a_3 - a_2^2 = \frac{1}{6} S_f(0)$, where $S_f(z) = (f''/f')' - \frac{1}{2}(f''/f')^2$ is Schwarzian derivative.

In this paper, we let S be the class of all univalent functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A classical theorem of Fekete and Szegö states:

Theorem A. ([7]) If $f \in S$ and $\lambda \in \mathbb{R}$, then

$$\max_{f\in S} |a_3 - \lambda a_2^2| = \begin{cases} 3 - 4\lambda, & \lambda < 0; \\ 1 + 2e^{-2\lambda/(1-\lambda)}, & 0 \le \lambda < 1; \\ 4\lambda - 3, & \lambda \ge 1. \end{cases}$$

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Since the importance of the Fekete-Szegö problem, the studies of this problem on various subclasses of S were widely concerned by many mathematicians, such as strongly starlike function of order α, close-to-convex function and so on [8,12,13]. Furthermore, Xu et al. [17] consider the Fekete-Szegö problem on $\mathbb{C}^n$, Elin et al. [5, 6] consider the "inverse Fekete-Szegö problem" which is related to infinitesimal generators.

For $0 \leq k < 1$, the subclass S_k formed by all $f \in S$ admitting k-quasiconformal extension to the complex plane $\mathbb{C}$. As we know S_k has attracted widely attention, since it is a classical topic in Geometric Function Theory and closely related to Teichmüller theory [14]. In my knowledge background, it is difficult to study the Fekete-Szegö problem on S_k because of the lack of equivalent characterization of functions in S_k. In 1972 and 1992, by Löwner chain theories, Becker [3, 4] and Betker [2] discovered remarkable facts, which give sufficient conditions for $f \in S_k$. Here, we introduce some basic concepts about Löwner chain theories, for more details, we refer to [16].

Definition 1.1. A function $f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ defined on $\mathbb{D} \times [0, \infty)$ is called a Löwner chain if $f_t(z) = f(z,t)$ is holomorphic and univalent in $\mathbb{D}$ for each $t \in [0, \infty)$ and satisfy $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$, for $0 \leq s < t < \infty$.

By Löwner theory, a Löwner chain $f(z,t)$ satisfies a equation

$$\frac{\partial f(z,t)}{\partial t} = z \frac{\partial f(z,t)}{\partial z} p(z,t), \ z \in \mathbb{D}, \ t \in [0, \infty), \tag{1.1}$$

where $p(z,t)$ is Herglotz function i.e., $p(z, \cdot)$ is measurable for each $t \in [0, \infty)$, $p(\cdot, t)$ is holomorphic in $\mathbb{D}$, satisfies $\mathrm{Re}p(\cdot, t) > 0$ for a.e. $t \geq 0$ and $p(0,t) = 1$. Moreover, given $f \in S$, there exist a Löwner chain $f(z,t)$ such that $f(z,0) = f(z)$. Conversely, given a Herglotz function from Löwner-Kufarev equation, we can obtain a Löwner chain.

Theorem B. ([16]) For any Herglotz function $p(z,t)$, the Löwner-Kufarev equation

$$\begin{cases} \frac{dw}{dt} = -wp(w,t), & t \geq s > 0; \\ w(s) = z, & z \in \mathbb{D}, \end{cases}$$

has a unique solution $w(t) = \varphi(z,s,t)$, the function $\varphi(z,s,t)$ are univalent in $z \in \mathbb{D}$, and

$$f(z,s) = \lim_{t \to \infty} e^t \varphi(z,s,t)$$

exists locally uniformly in $z \in \mathbb{D}$ and is a Löwner chain satisfying (1.1).

With the above preparations, we will now introduce the results of Becker and Betker.

Theorem C. ([3] [4]) Suppose that $f(z,t)$ is a Löwner chain with Herglotz function $p(z,t)$ satisfy

$$|\frac{p(z,t)-1}{p(z,t)+1}| \leq k,\ z \in \mathbb{D},\ a.e.\ \ t \in [0,\infty).$$

Then $f(z)=f(z,0)$ admitting k-quasiconformal extension on $\mathbb{C}$.

Theorem D. ([2]) Suppose that $f(z,t)$ is a Löwner chain with Herglotz function $p(z,t)$ satisfy

$$|\frac{p(z,t)-\overline{q(z,t)}}{p(z,t)+q(z,t)}| \leq k,\ z \in \mathbb{D},\ a.e.\ \ t \in [0,\infty),$$

where $q(z,t)$ is analytic $z \in \mathbb{D}$ and measurable in $t>0$ with $\mathrm{Re}q(z,t)>0$. Then $f(z)=f(z,0)$ admitting k-quasiconformal extension to $\mathbb{C}$.

Remark 1.1. When $q(z,t)=1$, Theorem D is just Theorem C. When $q\left(z,t\right)=p\left(z,t\right)$, then condition becomes $|argp\left(z,t\right)| \leq \arcsin\, k$.

Using Theorem C and Theorem D, we define two subclasses. We say $f \in S_k^B$, if $f \in S$ admitting a Löwner chain with Herglotz function $p(z,t)$ satisfy

$$p\left(\mathbb{D},t\right) \subseteq U\left(k\right) := \left\{w \in \mathbb{C} : |\frac{w-1}{w+1}| \leq k\right\},\ a.e.\ t \geq 0, \tag{1.2}$$

and $f \in S_k^{BT}$, if $f \in S$ admitting a Löwner chain with Herglotz function $p(z,t)$ satisfy

$$p\left(\mathbb{D},t\right) \subseteq A\left(k\right) := \left\{w \in \mathbb{C} : |arg\ w| \leq \arcsin\, k\right\},\ a.e.\ t \geq 0. \tag{1.3}$$

From Theorem C and Theorem D, we have $S_k^B \subset S_k$ and $S_k^{BT} \subset S_k$. In this article, we consider the Fekete-Szegö problem on S_k^B and S_k^{BT} and obtain the following results.

Theorem 1. For $0 \leq k < 1$, let $f(z)=z+a_2z^2+a_3z^3+... \in S_k^B$. Then, for $\lambda \in \mathbb{R}$,

$$\max_{f \in S_k^B} |a_3-\lambda a_2^2| = \begin{cases} k^2(3-4\lambda), & \lambda < \frac{k-1}{2k}; \\ k+k\left(1+k\right)e^{-\frac{1-k+2k\lambda}{k(1-\lambda)}}, & \frac{k-1}{2k} \leq \lambda < 1; \\ k+k\left(1-k\right)e^{-\frac{1+k-2k\lambda}{k(\lambda-1)}}, & 1 \leq \lambda < \frac{k+1}{2k}; \\ k^2\left(4\lambda-3\right), & \lambda \geq \frac{k+1}{2k}. \end{cases}$$

Theorem 2. For $0 \leq k < 1$, let $f(z)=z+a_2z^2+a_3z^3+... \in S_k^{BT}$. Then, for $\lambda \in \mathbb{R}$,

$$\max_{f \in S_k^{BT}} |a_3-\lambda a_2^2| = \begin{cases} \alpha^2(3-4\lambda), & \lambda < \frac{\alpha-1}{2\alpha}; \\ \alpha+\alpha\left(1+\alpha\right)e^{-\frac{1-\alpha+2\alpha\lambda}{\alpha(1-\lambda)}}, & \frac{\alpha-1}{2\alpha} \leq \lambda < 1; \\ \alpha+\alpha\left(1-\alpha\right)e^{-\frac{1+\alpha-2\alpha\lambda}{\alpha(\lambda-1)}}, & 1 \leq \lambda < \frac{\alpha+1}{2\alpha}; \\ \alpha^2\left(4\lambda-3\right), & \lambda \geq \frac{\alpha+1}{2\alpha}, \end{cases}$$

where $\alpha=\frac{2}{\pi}\arcsin\, k$.

Remark 1.2. In Theorem 1, let $\lambda = 0$ then we obtain the third coefficient estimate

$$\max_{f \in S_k^B} |a_3| = k + k\,(k+1)\,e^{1-\frac{1}{k}}.$$

Which is exactly Theorem 3.1 of Gumenyuk and Hotta [9]. Theorem A is just the case of Theorem 1 and Theorem 2 when $k \to 1$.

2 Properties of Herglotz Functions

From introduction, we know Herglotz function plays a significance role in Löwner theories. In this section, we will give some properties of Herglotz functions which will be used in the proofs of Theorems 1 and 2.

The following lemma characterizes the relationship between the modulus of derivatives and the modulus of functions, we refer to [1, p.136] for the proof.

Lemma 2.1. *If f is holomorphic in $\mathbb{D}$ and satisfies the condition $|f(z)| \leq 1$, then*

$$|f'(z)| \leq \frac{1 - |f\,(z)\,|^2}{1 - |z|^2}.$$

Lemma 2.2. *Let $f(z,t)$ be a Löwner chain with Herglotz function $p(z,t) = 1 + c_1\,(t)\,z + c_2\,(t)\,z^2 + ...$ satisfy (1.2). Then*

$$|c_1(t)| \leq 2k, \tag{2.1}$$

$$2k + Rec_2(t) \geq \frac{1+k}{2k}(Rec_1(t))^2, \tag{2.2}$$

$$2k + Rec_2(t) \geq \frac{1-k}{2k}(Imc_1(t))^2. \tag{2.3}$$

Proof. We let

$$L(z) = \frac{(1-k) - (1+k)z}{(1-k)z - (1+k)},$$

it is easily to know that $L(z) : U(k) \to \mathbb{H} = \{w : Re\ w > 0\}$ and $L(1) = 1$. Furthermore, we let

$$\varphi(z,t) = \frac{1}{z}\frac{L(p(z,t)) - 1}{L(p(z,t)) + 1} = \frac{1}{2k}c_1(t) + \left(\frac{1}{2k}c_2(t) - \frac{1}{4k}(c_1\,(t))^2\right) z + \ldots,$$

then $\varphi(\cdot, t)$ is holomorphic in $\mathbb{D}$ and $|\varphi(z,t)| \leq 1$ for a.e. $t \geq 0$. Then (2.1) follows by

$$|\frac{1}{2k}c_1\,(t)\,| = |\varphi\,(0,t)\,| \leq 1.$$

Furthermore, using Lemma 2.1 for $\varphi(z,t)$ at $z = 0$, we obtain

$$|\frac{1}{2k}c_2(t) - \frac{1}{4k}(c_1(t))^2| = |\varphi'\,(0,t)\,| \leq 1 - |\varphi\,(0,t)\,|^2 = 1 - \frac{1}{4k^2}|c_1\,(t)\,|^2. \tag{2.4}$$

Taking the negative real part on the left-hand side of (2.4), we obtain

$$\begin{aligned}1-\frac{1}{4k^2}\left[Rec_1\left(t\right)\right]^2-\frac{1}{4k^2}\left[Imc_1(t)\right]^2 =&1-\frac{1}{4k^2}|c_1\left(t\right)|^2\\ \geq&|\frac{1}{2k}c_2(t)-\frac{1}{4k}(c_1(t))^2|\\ \geq&-Re\frac{1}{2k}c_2\left(t\right)+\frac{1}{4k}Re\left[c_1\left(t\right)\right]^2\\ =&-Re\frac{1}{2k}c_2\left(t\right)+\frac{1}{4k}\left[Rec_1\left(t\right)\right]^2-\frac{1}{4k}\left[Imc_1(t)\right]^2.\end{aligned}$$

It follows that

$$2k+Rec_2\left(t\right)\geq\frac{1+k}{2k}\left[Rec_1\left(t\right)\right]^2+\frac{1-k}{2k}\left[Imc_1(t)\right]^2,$$

which implies (2.2) and (2.3). □

Lemma 2.3. *Let $f(z,t)$ be a Löwner chain with Herglotz function $p(z,t)=1+c_1\left(t\right)z+c_2\left(t\right)z^2+...$ satisfy (1.3). Then*

$$|c_1(t)|\leq 2\alpha, \tag{2.5}$$

$$2\alpha+Rec_2(t)\geq\frac{1+\alpha}{2\alpha}(Rec_1(t))^2, \tag{2.6}$$

$$2\alpha+Rec_2(t)\geq\frac{1-\alpha}{2\alpha}(Imc_1(t))^2, \tag{2.7}$$

where $\alpha=\frac{2}{\pi}\arcsin k$.

Proof. In fact, the proof of Lemma 2.3 can be deduced step by step as the proof of Lemma 2.2 by define

$$\varphi(z,t)=\frac{1}{z}\frac{(p(z,t))^{\frac{1}{\alpha}}-1}{(p(z,t))^{\frac{1}{\alpha}}+1}=\frac{1}{2\alpha}c_1(t)+\left(\frac{1}{2\alpha}c_2(t)-\frac{1}{4\alpha}(c_1\left(t\right))^2\right)z+\ldots.$$

Here, we omit the details. □

By using (1.2), we obtain the relationships between the Taylor coefficients of $f\in S$ and the Taylor coefficients of Herglotz function $p(z,t)$. For details, we refer to [16, p.165] for the proof.

Lemma 2.4. ([16]) *Let $f(z,t)$ be a Löwner chain with Herglotz function $p(z,t)=1+c_1(t)z+c_2(t)z^2+...$ and $f(z,0)=f(z)=z+a_2z^2+a_3z^3+...\in S$. Then*

$$\begin{cases}a_2=-\int_0^\infty e^{-t}c_1\left(t\right)dt;\\ a_3=-\int_0^\infty e^{-2t}c_2\left(t\right)dt+\left(\int_0^\infty e^{-t}c_1\left(t\right)dt\right)^2.\end{cases}$$

3 Proofs of Theorems 1 and 2

In fact, to prove Theorems 1 and 2, we only need to consider $f \in S_k^B$ and $f \in S_k^{BT}$ such that $a_3 - \lambda a_2^2 \geq 0$ by the following lemmas.

Lemma 3.1. *Given $f(z) \in S_k^B$, then $e^{-i\theta} f(e^{i\theta} z) \in S_k^B$, for any $\theta \in [0, 2\pi)$.*

Proof. By the definition of S_k^B, for $f(z) \in S_k^B$, there is a Löwner chain $f(z,t)$ with Herglotz function $p(z,t)$ satisfy (1.2) and $f(z,0) = f(z)$.

$$\widetilde{f}(z,t) = e^{-i\theta} f\left(e^{i\theta} z, t\right),$$

it is easily to know $\widetilde{f}(z,t)$ is a Löwner chain and $\widetilde{f}(z,0) = e^{-i\theta} f\left(e^{i\theta} z\right)$, then we denote its corresponding Herglotz function by $\widetilde{p}(z,t)$. By using (1.1), we obtain

$$\widetilde{p}(z,t) = \frac{\partial \widetilde{f}(z,t)}{\partial t} / \left(z \frac{\partial \widetilde{f}(z,t)}{\partial z} \right) = e^{-i\theta} \frac{\partial f\left(e^{i\theta} z, t\right)}{\partial t} / \left(z \frac{\partial f\left(e^{i\theta} z, t\right)}{\partial z} \right) = p\left(e^{i\theta} z, t\right),$$

which implies $\widetilde{p}(z,t)$ satisfy (1.2). It follows that $e^{-i\theta} f(e^{i\theta} z) = z + a_2 e^{i\theta} z^2 + a_3 e^{2i\theta} z^3 + ... \in S_k^B$. □

Lemma 3.2. *Given $f(z) \in S_k^{BT}$, then $e^{-i\theta} f(e^{i\theta} z) \in S_k^{BT}$, for any $\theta \in [0, 2\pi)$.*

Proof. The proof of Lemma 3.2 can be deduced step by step as the proof of Lemma 3.1, thus we omit the proof. □

Following Lemmas 3.1 and 3.2, for simplicity, we can assume $a_3 - \lambda a_2^2 \geq 0$ for $f \in S_k^B$ or $f \in S_k^{BT}$. From Lemma 2.4, we obtain

$$a_3 - \lambda a_2^2 = -\int_0^\infty e^{-2t} Re c_2(t)\, dt + (1-\lambda) \left(\int_0^\infty e^{-t} Re c_1(t)\, dt \right)^2 - (1-\lambda) \left(\int_0^\infty e^{-t} Im c_1(t)\, dt \right)^2, \quad (3.1)$$

we denote $Re c_1(t) = u(t)$ and $Im c_1(t) = v(t)$.

In the following, we will give the proof of Theorems 1 and 2. By using the properties of the Herglotz function in Section 2, we will obtain the upper bound for the Fekete-Szegö problem, and by taking the appropriate Herglotz functions and using Theorem B, we can find the extreme functions for the Fekete-Szegö problem.

Proof of Theorem 1. Given $f(z) \in S_k^B$, we divide $\lambda \in \mathbb{R}$ into four cases and prove the Fekete-Szegö problem in these cases respectively.

Case I: For $\lambda < \frac{2k-1}{2k}$, we prove $|a_3 - \lambda a_2^2| \leq k^2 (3 - 4\lambda)$ and then we find a function $\widetilde{f}(z) \in S_k^B$ such that equality holds.

By using (2.1), (2.2) and (3.1), we get the following estimate

$$\begin{aligned}
a_3-\lambda a_2^2 \leq & -\int_0^\infty e^{-2t}Re\ c_2(t)\,dt+(1-\lambda)\left(\int_0^\infty e^{-t}u(t)\,dt\right)^2 \\
\leq & \int_0^\infty e^{-2t}[2k-\frac{k+1}{2k}u^2(t)]dt+(1-\lambda)\left(\int_0^\infty e^{-t}u(t)dt\right)^2 \\
= & k-\int_0^\infty \frac{k+1}{2k}e^{-2t}u^2(t)dt+(1-\lambda)\left(\int_0^\infty e^{-t}\frac{u(t)}{e^{-\frac{t}{2}}}e^{-\frac{t}{2}}dt\right)^2 \\
\leq & k-\int_0^\infty \frac{k+1}{2k}e^{-2t}u^2(t)dt+(1-\lambda)\int_0^\infty e^{-t}u^2(t)\,dt \\
= & k+\int_0^\infty e^{-2t}u^2(t)\left\{(1-\lambda)e^t-\frac{k+1}{2k}\right\}dt \\
\leq & k+4k^2\int_0^\infty (1-\lambda)e^{-t}-\frac{k+1}{2k}e^{-2t}dt \\
= & k+4k^2(1-\lambda)-k(k+1) \\
= & k^2(3-4\lambda).
\end{aligned}$$

The first inequality can be obtained directly from (3.1), the second inequality can be obtained from (2.2), the third inequality holds since Schwarz's inequality and the last inequality holds since $|u(t)|\leq 2k$ and $\lambda<\frac{k-1}{2k}$.

Next, we show for every λ the equality is possible. We Let

$$p(z,t)=\frac{1-kz}{1+kz}=1-2kz+2k^2z^2+...,\ t\geq 0.$$

It is easily know $p(z,t)$ satisfy (1.2). By using Theorem B, we can find a Löwner chain $f(z,t)$, furthermore $\widetilde{f}(z)=f(z,0)\in S_k^B$. We denote $\widetilde{f}(z)=z+\widetilde{a_2}z^2+\widetilde{a_3}z^3+...$ and use (3.1), then it follows that

$$\widetilde{a_3}-\lambda\widetilde{a_2}^2=-\int_0^\infty e^{-2t}2k^2dt+(1-\lambda)\left(\int_0^\infty e^{-t}2kdt\right)^2=k^2(3-4\lambda).$$

Which implies $\widetilde{f}(z)$ is the extreme function for the Fekete-Szegö problem on S_k^B, for $\lambda<\frac{k-1}{2k}$.

Case II: For $\frac{2k-1}{2k}\leq\lambda<1$, we prove $|a_3-\lambda a_2^2|<k+k(1+k)e^{-\frac{1-k+2k\lambda}{k(1-\lambda)}}$ and then we find a function $\widetilde{f}(z)\in S_k^B$ such that equality holds.

By using (2.2) and(3.1), we obtain

$$a_3-\lambda a_2^2\leq\int_0^\infty e^{-2t}[2k-\frac{k+1}{2k}u^2(t)]dt+(1-\lambda)\left(\int_0^\infty e^{-t}u(t)dt\right)^2.$$

We estimate the second integral by Schwarz's inequality. For every positive continuous function $H(t)$ we obtain

$$a_3-\lambda a_2^2\leq k+\int_0^\infty e^{-2t}u^2(t)\Big[-\frac{1+k}{2k}+\frac{(1-\lambda)b}{H(t)}\Big]dt, \tag{3.2}$$

where $b = \int_0^\infty H(t)\,dt$, then the best estimate is obtained if the integral vanish on some initial rang $[0,\tau]$, and we choose

$$H(t) = \begin{cases} \frac{2k}{1+k}(1-\lambda)\,b, & 0 \le t \le \tau; \\ \frac{2k}{1+k}(1-\lambda)be^{\tau-t}, & \tau \le t < \infty. \end{cases}$$

By the definition of b, we can obtain $b = \frac{2k}{k+1}(1-\lambda)b(\tau+1)$ and therefore $\tau = \frac{\frac{1-k}{2k}+\lambda}{1-\lambda} > 0$. By using (2.1), we conclude from (3.2)

$$a_3 - \lambda a_2^2 \le k + 2k(1+k)\int_\tau^\infty e^{-2t}\left(e^{t-\tau}-1\right)dt = k + (k+1)\,e^{-\frac{1-k+2k\lambda}{(1-\lambda)k}}.$$

Next we show for every λ the equality is possible. Let $\tau = \frac{\frac{1-k}{2k}+\lambda}{1-\lambda}$ and consider function

$$p(z,t) = \begin{cases} \frac{1-kz^2+(1-k)e^{t-\tau}z}{1+kz^2+(1+k)e^{t-\tau}z} = 1 - 2ke^{t-\tau}z + \left\{2k(k+1)e^{2t-2\tau} - 2k\right\}z^2 + \ldots, & 0 \le t < \tau; \\ \frac{1-kz}{1+kz} = 1 - 2kz + 2k^2z^2 + \ldots, & \tau \le t < \infty. \end{cases}$$

For $0 \le t < \tau$, since

$$\left|\frac{p(z,t)-1}{p(z,t)+1}\right| = \left|\frac{z^2+e^{t-\tau}z}{1+e^{t-\tau}z}\right|k \le k,$$

then $p(z,t)$ satisfy (1.2). By using Theorem B, we can find a Löwner chain $f(z,t)$, then $\widetilde{f}(z) = f(z,0) \in S_k^B$. Denoting $\widetilde{f}(z) = z + \widetilde{a_2}z^2 + \widetilde{a_3}z^3 + \ldots$ and using (3.1), we immediately obtain

$$\begin{aligned} \tilde{a_3} - \lambda\tilde{a_2}^2 &= -\int_0^\tau e^{-2t}\left\{2k(k+1)e^{2t-2\tau} - 2k\right\}dt - \int_\tau^\infty e^{-2t}2k^2dt \\ &+ (1-\lambda)\left(\int_0^\tau -2ke^{t-\tau}e^{-t}dt + \int_\tau^\infty -2ke^{-t}dt\right)^2 \\ &= k + \{-2k(k+1)\tau - k - k^2 + (1-\lambda)4k^2(\tau+1)^2\}e^{-2\tau} \\ &= k + \left\{\frac{k^2-1-2k(k+1)\lambda}{1-\lambda} + \frac{(1+k)^2}{1-\lambda} - k - k^2\right\}e^{-2\tau} \\ &= k + \left\{\frac{2k(1+k)-2k(1+k)\lambda}{1-\lambda} - k - k^2\right\}e^{-2\tau} \\ &= k + k(k+1)e^{-2\tau} = k + k(k+1)e^{-\frac{1-k+2k\lambda}{k(1-\lambda)}}. \end{aligned}$$

Which shows $\widetilde{f}(z)$ is the extreme function for the Fekete-Szegö problem, for $\frac{k-1}{2k} \le \lambda < 1$.

Case III: For $1 \le \lambda < \frac{k+1}{2k}$, we prove $|a_3 - \lambda a_2^2| < k + k(1-k)e^{-\frac{1+k-2k\lambda}{k(\lambda-1)}}$ and then we find a function $\widetilde{f}(z) \in S_k^B$ such that equality holds.

By using (2.3) and (3.1) and get the following estimate

$$a_3 - \lambda a_2^2 \le \int_0^\infty e^{-2t}\left[2k - \frac{1-k}{2k}v^2(t)\right]dt - (1-\lambda)\left(\int_0^\infty e^{-t}v(t)\,dt\right)^2.$$

We estimate the second integral by Schwarz's inequality, for every positive continuous function $H(t)$ we obtain

$$a_3-\lambda a_2^2\le k+\int_0^{\infty}e^{-2t}v^2\left(t\right)\Big[-\frac{1-k}{2k}+\frac{(\lambda-1)b}{H\left(t\right)}\Big]dt, \tag{3.3}$$

where $b=\int_0^{\infty}H\left(t\right)dt$. Then we choose

$$H(t)=\begin{cases}\frac{2k}{1-k}\left(\lambda-1\right)b, & 0\le t\le\tau;\\ \frac{2k}{1-k}(\lambda-1)be^{\tau-t}, & \tau\le t<\infty,\end{cases}$$

by the definition of b we can obtain $b=\frac{2k}{1-k}(\lambda-1)b\left(\tau+1\right)$ and therefore $\tau=\frac{\frac{1+k}{2k}-\lambda}{\lambda-1}>0$. Since $|v(t)|=|Imc_1(t)|\le 2k$, we conclude from (3.3)

$$a_3-\lambda a_2^2\le k+2k(1-k)\int_{\tau}^{\infty}e^{-2t}\left(e^{t-\tau}-1\right)dt=k+k\left(1-k\right)e^{-\frac{1+k-2k\lambda}{k(\lambda-1)}}.$$

Next we show for every λ the equality is possible. Let $\tau=\frac{\frac{1+k}{2k}-\lambda}{\lambda-1}$ and consider function

$$p(z,t)=\begin{cases}\frac{1-kz^2+(1+k)e^{t-\tau}iz}{1+kz^2+(1-k)e^{t-\tau}iz}=1+2ke^{t-\tau}iz+\left\{2k(1-k)e^{2t-2\tau}-2k\right\}z^2+\dots, & 0\le t<\tau;\\ \frac{1+ikz}{1-ikz}=1+2kiz-2k^2z^2+\dots, & \tau\le t<\infty.\end{cases}$$

For $0\le t<\tau$, since

$$|\frac{p\left(z,t\right)-1}{p\left(z,t\right)+1}|=|\frac{-z^2+e^{t-\tau}iz}{1+e^{t-\tau}iz}|k=|\frac{\left(iz\right)^2+e^{t-\tau}iz}{1+e^{t-\tau}iz}|k\le k,$$

it follows that $p(z,t)$ satisfy (1.2). By using Theorem B, we can find a extreme function $\widetilde{f}(z)\in S_k^B$ for the Fekete-Szegö problem. We denote $\widetilde{f(z)}=z+\widetilde{a}_2z^2+\widetilde{a}_3z^3+...$ and use (3.1), then we obtain

$$\begin{aligned}\widetilde{a}_3-\lambda\widetilde{a}_2{}^2&=-\int_0^{\tau}e^{-2t}\left\{2k\left(1-k\right)e^{2t-2\tau}-2k\right\}dt+\int_{\tau}^{\infty}e^{-2t}2k^2dt\\&+\left(\lambda-1\right)\left(\int_0^{\tau}2ke^{t-\tau}e^{-t}dt+\int_{\tau}^{\infty}2ke^{-t}dt\right)^2\\&=k+\{-2k\left(1-k\right)\tau-k+k^2+\left(\lambda-1\right)4k^2\left(\tau+1\right)^2\}e^{-2\tau}\\&=k+\left\{\frac{k^2-1+2k\left(1-k\right)\lambda}{\lambda-1}+\frac{\left(1-k\right)^2}{\lambda-1}-k+k^2\right\}e^{-2\tau}\\&=k+\left\{\frac{-2k\left(1-k\right)+2k\left(1-k\right)\lambda}{\lambda-1}-k+k^2\right\}e^{-2\tau}\\&=k+k\left(1-k\right)e^{-\frac{1+k-2k\lambda}{k(\lambda-1)}}.\end{aligned}$$

Case IV: For $\lambda\ge\frac{2k+1}{2k}$, we prove $|a_3-\lambda a_2^2|\le k^2\left(4\lambda-3\right)$ and then we find a function $\widetilde{f}(z)\in S_k^B$ such that equality holds.

By using (2.1), (2.3) and (3.1), we obtain the following estimate

$$\begin{aligned}
a_3-\lambda a_2^2 \leq & -\int_0^{\infty} e^{-2t} Re\ c_2(t)\,dt-(1-\lambda)\left(\int_0^{\infty} e^{-t} v(t)\,dt\right)^2 \\
\leq & \int_0^{\infty} e^{-2t}[2k-\frac{1-k}{2k}v^2(t)]dt-(1-\lambda)\left(\int_0^{\infty} e^{-t} v(t) dt\right)^2 \\
= & k-\int_0^{\infty} \frac{1-k}{2k} e^{-2t} v^2(t) dt-(1-\lambda)\left(\int_0^{\infty} e^{-t} \frac{v(t)}{e^{-\frac{t}{2}}} e^{-\frac{t}{2}} dt\right)^2 \\
\leq & k-\int_0^{\infty} \frac{1-k}{2k} e^{-2t} v^2(t) dt-(1-\lambda) \int_0^{\infty} e^{-t} v^2(t)\,dt \\
= & k+\int_0^{\infty} e^{-2t} v^2(t)\left\{(\lambda-1) e^t-\frac{1-k}{2k}\right\} dt \\
\leq & k+4k^2 \int_0^{\infty} e^{-t}(\lambda-1)-\frac{1-k}{2k} e^{-2t} dt \\
= & k+4k^2(\lambda-1)-k(1-k) \\
= & k^2(4\lambda-3).
\end{aligned}$$

The first inequality holds since $1-\lambda \leq 0$, the second inequality can be obtained from (2.3), the third inequality holds since Schwarz's inequality and the last inequality holds since $|v(t)| \leq 2k$ and $\lambda < \frac{k-1}{2k}$.

Next we show for every λ the equality is possible. We let

$$p(z,t)=\frac{1-ikz}{1+ikz}=1-2ikz-2k^2z^2+\ldots,\ t \geq 0.$$

It is easily to check $p(z,t)$ satisfy (1.6). Then, by using Theorem B, we can find a function $\widetilde{f}(z) \in S_k^B$. Then directly calculate from (3.1) we obtain

$$\widetilde{a_3}-\lambda \widetilde{a_2}^2=\int_0^{\infty} e^{-2t} 2k^2 dt-(1-\lambda)\left(\int_0^{\infty} e^{-t} 2k dt\right)^2=k^2+(\lambda-1)4k^2=k^2(4\lambda-3).$$

where $\widetilde{a_2}$, $\widetilde{a_3}$ are Taylor coefficients of $\widetilde{f}(z)$. □

In fact, using Lemma 2.3 and (3.1), the upper bound of the Fekete-Szegö problem on S_k^{BT} can be deduced step by step as the proof of Theorem 1, thus we omit the details. Here, we only show when equality holds.

Proof of Theorem 2. Given $f \in S_k^{BT}$, we divide $\lambda \in \mathbb{R}$ into four cases and prove the Fekete-Szegö problem in these cases respectively.

Case I: For $\lambda < \frac{2\alpha-1}{2\alpha}$, we find a extreme function $\widetilde{f}(z) \in S_k^{\mathrm{BT}}$ such that $|\widetilde{a_3}-\lambda \widetilde{a}_2^2| = \alpha^2(3-4\lambda)$.

Let Herglotz function be

$$p(z,t)=\left(\frac{1-z}{1+z}\right)^{\alpha}=1-2\alpha z+2\alpha^2z^2+...,\ t\geq 0.$$

It is easily know that $p(z,t)$ satisfy (1.3). Using Theorem B, we can find a extreme function $\widetilde{f}(z)\in S_k^{BT}$ Then by (3.1) we obtain

$$\widetilde{a_3}-\lambda\widetilde{a_2}^2=-\int_0^{\infty}e^{-2t}2\alpha^2dt+(1-\lambda)\left(\int_0^{\infty}e^{-t}2\alpha dt\right)^2=\alpha^2\left(3-4\lambda\right).$$

Case II: For $\frac{\alpha-1}{2\alpha}\leq\lambda<1$, we find a extreme function $\widetilde{f}(z)\in S_k^{\mathrm{BT}}$ such that $|\widetilde{a_3}-\lambda\widetilde{a}_2^2|$ $=\alpha+\alpha\left(1+\alpha\right)e^{-\frac{1-\alpha+2\alpha\lambda}{\alpha(1-\lambda)}}$.

We let Herglotz function be

$$p(z,t)=\begin{cases}\left(\frac{1-z^2}{1+z^2+2e^{t-\tau}z}\right)^{\alpha}=1-2\alpha e^{t-\tau}z+\left\{2\alpha\left(\alpha+1\right)e^{2t-2\tau}-2\alpha\right\}z^2+..., & 0\leq t<\tau;\\ \left(\frac{1-z}{1+z}\right)^{\alpha}=1-2\alpha z+2\alpha^2z^2+..., & t\geq\tau,\end{cases}$$

where $\tau=\frac{\frac{1-\alpha}{2\alpha}+\lambda}{1-\lambda}$. For $0\leq t<\tau$, since

$$Re\frac{1+z^2+2e^{t-\tau}z}{1-z^2}=Re\left\{\frac{1+z^2}{1-z^2}+2e^{t-\tau}\frac{z}{1-z^2}\right\}\geq 0,$$

it follows that $\mathrm{Re}\frac{1-z^2}{1+z^2+2e^{t-\tau}z}>0$ and $p(z,t)$ satisfy (1.3).

By using Theorem B, we can find a extreme function $\widetilde{f}(z)\in S_k^{BT}$, let $\widetilde{a_2}$, $\widetilde{a_3}$ be the Taylor coefficients of $\widetilde{f}(z)$. Then directly calculate from (3.1) we obtain

$$\begin{aligned}\widetilde{a_3}-\lambda\widetilde{a_2}^2&=-\int_0^{\tau}e^{-2t}\left\{2\alpha\left(\alpha+1\right)e^{2t-2\tau}-2\alpha\right\}dt-\int_{\tau}^{\infty}e^{-2t}2\alpha^2dt\\&\quad+(1-\lambda)\left(\int_0^{\tau}-2\alpha e^{t-\tau}e^{-t}dt+\int_{\tau}^{\infty}-2\alpha e^{-t}dt\right)^2\\&=\alpha+\{-2\alpha\left(\alpha+1\right)\tau-\alpha-\alpha^2+(1-\lambda)\,4\alpha^2\left(\tau+1\right)^2\}e^{-2\tau}\\&=\alpha+\left\{\frac{\alpha^2-1-2\alpha\left(\alpha+1\right)\lambda}{1-\lambda}+\frac{(1+\alpha)^2}{1-\lambda}-\alpha-\alpha^2\right\}e^{-2\tau}\\&=\alpha+\left\{\frac{2\alpha\left(1+\alpha\right)-2\alpha\left(1+\alpha\right)\lambda}{1-\lambda}-\alpha-\alpha^2\right\}e^{-2\tau}\\&=\alpha+\alpha\left(\alpha+1\right)e^{-2\tau}=\alpha+\alpha\left(\alpha+1\right)e^{-\frac{1-\alpha+2\alpha\lambda}{\alpha(1-\lambda)}}.\end{aligned}$$

Case III: For $1\leq\lambda<\frac{\alpha+1}{2\alpha}$, we find a extreme function $\widetilde{f}(z)\in S_k^{\mathrm{BT}}$ such that $|\widetilde{a_3}-\lambda\widetilde{a}_2^2|$ $=\alpha+\alpha\left(1-\alpha\right)e^{-\frac{1+\alpha-2\alpha\lambda}{\alpha(\lambda-1)}}$

We let Herglotz function be

$$p(z,t)=\begin{cases}\left(\frac{1-z^2+2e^{t-\tau}iz}{1+z^2}\right)^{\alpha}=1+2\alpha e^{t-\tau}iz+\left\{2\alpha(1-\alpha)e^{2t-2\tau}-2\alpha\right\}z^2+..., & 0\le t<\tau;\\ \left(\frac{1+iz}{1-iz}\right)^{\alpha}=1+2\alpha iz-2\alpha^2z^2+..., & t\ge\tau,\end{cases}$$

where $\tau=\frac{\frac{1+\alpha}{2\alpha}-\lambda}{\lambda-1}$. For $0\le t<\tau$, since

$$Re\frac{1-z^2+2e^{t-\tau}iz}{1+z^2}=Re\frac{1-z^2}{1+z^2}+2e^{t-\tau}Re\frac{iz}{1-(iz)^2}>0,$$

which implies $p(z,t)$ satisfies (1.3). By using Theorem B, we can find a extreme function $\widetilde{f}(z)\in S_k^{BT}$, let $\widetilde{a_2}$, $\widetilde{a_3}$ be the Taylor coefficients of $\widetilde{f}(z)$. Then

$$\begin{aligned}\widetilde{a_3}-\lambda\widetilde{a_2}^2&=-\int_0^{\tau}e^{-2t}\left\{2\alpha(1-\alpha)e^{2t-2\tau}-2\alpha\right\}dt+\int_{\tau}^{\infty}e^{-2t}2\alpha^2dt\\&+(\lambda-1)\left(\int_0^{\tau}2\alpha e^{t-\tau}e^{-t}dt+\int_{\tau}^{\infty}2\alpha e^{-t}dt\right)^2\\&=\alpha+\{-2\alpha(1-\alpha)\tau-\alpha+\alpha^2+(\lambda-1)4\alpha^2(\tau+1)^2\}e^{-2\tau}\\&=\alpha+\left\{\frac{\alpha^2-1+2\alpha(1-\alpha)\lambda}{\lambda-1}+\frac{(1-\alpha)^2}{\lambda-1}-\alpha+\alpha^2\right\}e^{-2\tau}\\&=\alpha+\left\{\frac{-2\alpha(1-\alpha)+2\alpha(1-\alpha)\lambda}{\lambda-1}-\alpha+\alpha^2\right\}e^{-2\tau}\\&=\alpha+\alpha(1-\alpha)e^{-\frac{1+\alpha-2\alpha\lambda}{\alpha(\lambda-1)}}.\end{aligned}$$

Case IV: For $\lambda\ge\frac{2\alpha+1}{2\alpha}$, we find a extreme function $\widetilde{f}(z)\in S_k^{\mathbf{BT}}$ such that $|a_3-\lambda a_2^2|=\alpha^2(4\lambda-3)$

Let Herglotz function be

$$p(z,t)=\left(\frac{1-iz}{1+iz}\right)^{\alpha}=1-2i\alpha z-2\alpha^2z^2+...\ ,\ t\ge0,$$

It is obviously that $p(z,t)$ satisfies (1.3). Then we can find a extreme function $\widetilde{f}(z)\in S_k^{BT}$, by using Theorem B. Let $\widetilde{a_2}$, $\widetilde{a_3}$ be the Taylor coefficients of $\widetilde{f}(z)$. Then

$$\widetilde{a_3}-\lambda\widetilde{a_2}^2=\int_0^{\infty}e^{-2t}2\alpha^2dt+(\lambda-1)\left(\int_0^{\infty}e^{-t}2\alpha dt\right)^2=\alpha^2(4\lambda-3).$$

□

References

[1] Ahlfors, L. (1996). *Complex analysis.* New York: McGraw-Hill.

[2] Betker, T. (1992). Löwner chains and quasiconformal extensions. *Complex Variables Theory Appl.*, *20*(1-4), 107-111. https://doi.org/10.1080/17476939208814591

[3] Becker, J. (1972). Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. (German) *J. Reine Angew. Math.*, *255*, 23-43. https://doi.org/10.1515/crll.1972.255.23

[4] Becker, J. (1980). Conformal mappings with quasiconformal extensions. In *Aspects of contemporary complex analysis* (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979), pp. 37-77, Academic Press, London-New York.

[5] Elin, M., & Jacobzon, F. (2022). The Fekete-Szegö problem for spirallike mappings and non-linear resolvents in Banach spaces. *Stud. Univ. Babeş-Bolyai Math.*, *67*(2), 329-344. https://doi.org/10.24193/subbmath.2022.2.09

[6] Elin, M., & Jacobzon, F. (2022). Note on the Fekete-Szegö problem for spirallike mappings in Banach spaces. *Results Math.*, *77*(3), Paper No. 137, 6 pp. https://doi.org/10.1007/s00025-022-01672-x

[7] Fekete, M., & Szegö, G. (1933). Eine Bermerkung über ungerade schlichte Funktionen. *J. London Math. Soc.*, *8*, 85-89. https://doi.org/10.1112/jlms/s1-8.2.85

[8] Gawad, H., & Thomas, D. (1992). The Fekete-Szegö problem for strongly close-to-convex functions. *Proc. Amer. Math. Soc.*, *114*(2), 345-349. https://doi.org/10.1090/S0002-9939-1992-1065939-0

[9] Gumenyuk, P., & Hotta, I. (2020). Univalent functions with quasiconformal extensions: Becker's class and estimates of the third coefficient. *Proc. Amer. Math. Soc.*, *148*(9), 3927-3942. https://doi.org/10.1090/proc/15010

[10] Hotta, I. (2019). Loewner theory for quasiconformal extensions: old and new. *Interdiscip. Inform. Sci.*, *25*(1), 1-21. https://doi.org/10.4036/iis.2019.A.01

[11] Hotta, I. (2009). Explicit quasiconformal extensions and Löwner chains. *Proc. Japan Acad. Ser. A Math. Sci.*, *85*(8), 108-111. https://doi.org/10.3792/pjaa.85.108

[12] Koepf, W. (1987). On the Fekete-Szegö problem for close-to-convex functions. II. *Arch. Math. (Basel)*, *49*(5), 420-433. https://doi.org/10.1007/BF01194100

[13] Keogh, R., & Merkes, E. (1969). A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, *20*, 8-12. https://doi.org/10.1090/S0002-9939-1969-0232926-9

[14] Lehto, O. (1987). *Univalent Functions and Teichmüller Spaces.* Graduate Texts in Mathematics, vol. 109, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4613-8652-0_3

[15] Ma, W., & Minda, D. (1991). An internal geometric characterization of strongly starlike functions. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, *45*, 89-97 (1992).

[16] Pommerenke, Ch. (1975). *Univalent functions*. Vandenhoeck and Ruprecht.

[17] Xu, Q., Fang, W., Feng, W., & Lui, T. (2023). The Fekete-Szegö inequality and successive coefficients difference for a subclass of close-to-starlike mappings in complex Banach spaces. *Acta Mathematica Scientia*, *43*(5), 2075-2088. https://doi.org/10.1007/s10473-023-0509-5

[18] Thomas, D., & Verma, S. (2017). Invariance of the coefficients of strongly convex functions. *Bulletin of the Australian Mathematical Society*, *95*(3), 436-445. https://doi.org/10.1017/S0004972716000976