Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 3, 2025, Pages 345-365
https://doi.org/10.34198/ejms.15325.345365

Block Hybrid Trapezoidal-type Methods for Solving Initial Value Problems in Ordinary Differential Equations

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Abstract

In solving ordinary differential equations, tackling stiff problems necessitates the application of robust numerical methods endowed with A-stability properties. To circumvent the constraints posed by the Dahlquist barrier theorem and mitigate errors arising from step-by-step implementation of linear multistep methods, block hybrid schemes have been introduced. This study focuses on the development of novel block schemes designed for the direct approximation of solutions to stiff initial value problems. The methods proposed herein leverage both interpolation and collocation, enhancing their consistency, convergence, and accuracy in solving initial value problems. The efficacy of the devised methods is demonstrated through a comprehensive analysis of stability regions for each of the constructed block algorithms. Notably, these stability regions are proven to be unbounded for order $p \leq 15$. Comparative assessments reveal their competitiveness with existing methods. In fact, this research introduces innovative approaches to address the challenges posed by stiff initial value problems, offering enhanced stability and accuracy in comparison to established methods.

1 Introduction

Ordinary differential equations (ODEs) are fundamental in modeling dynamic systems across various disciplines, including engineering, economics, social science, biology, and health science. Many of these models can be transformed into systems of ODEs, which can often be classified as stiff equations. These equations typically take the form:

$$y' = Ay, \quad y(\alpha) = y_0, \quad \alpha \leq x \leq \beta \tag{1.1}$$

Received: December 24, 2024; Revised & Accepted: February 3, 2025; Published: February 18, 2025
2020 Mathematics Subject Classification: Primary 65L05; Secondary 65L05.
Keywords and phrases: hybrid self-starting methods, block methods, stiff systems.
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with $y \in R^m$, the matrix A is an $m \times m$ with the roots $\{\lambda_i\}_{i=1}^{m}$ such that $Re(\lambda_i) < 0$. In line with Cash [1], the major requirement to implement equation in (1.1) is to have a good method coupled with high order and good stability conditions. However, the Dalquist [2] barrier conditions posses a restriction on the traditional linear multistep methods (LMMs). Due to this limitations, modification of LMMs was considered by Bickart and Rubin [3]. In the light of this, [4–10] developed special family of Obreckoff [11] multi-derivative methods. The concept of hybrid LMMs, which integrate off-step points into traditional LMMs, has been proposed by several authors, including [12–16]. These hybrid methods aim to bypass the limitations imposed by the Dahlquist order barrier and have shown promising results. The implementation of hybrid methods, however, is often more computationally intensive due to the additional steps involved see Gupta [17]. In this paper, we propose a self-starting hybrid scheme that does not require additional methods for its implementation. Derived through interpolation and collocation, this method is designed to be more accurate and possess a smaller error constant compared to conventional LMMs. Implemented as a block method, the proposed scheme demonstrates significant improvements in accuracy and efficiency. This paper is structured as follows. In Section 2 we derive and analyze the obtained schemes for consistency, zero stability, while the stability properties of the new methods is in Section 3. The numerical results achieved using the new schemes, along with those from existing methods, are presented in Section 4

2 Development of the New Methods

Consider the initial value problems of the form

$$y' = f(x, y). \tag{2.1}$$

Given a continuous hybrid method of the form

$$\sum_{j=0}^{k} \alpha_{i,j} y_{n+p_j} - h\alpha_{i,k+1} f_{n+p_{i-1}} = hf_{n+p_i}; \quad p_i = \frac{i}{k}, \quad i = 1, 2, \cdots, k; \quad k \geq 2, \tag{2.2}$$

on hybrid-step points $x_0, x_{\frac{1}{k}}, x_{\frac{2}{k}}, \cdots, x_{\frac{k-1}{k}}, x_1$. The y_{n+p_i} denotes the discrete approximation of the analytic solution $y(x_n + p_i)$ at $x_n + p_i$, $f_{n+p_i} = f(x_n + p_i, y_n + p_i)$, h is the given stepsize and the α_j are real coefficients that are determined through the hybrid point using interpolation and collocation approach. Thus, the method in (2.2) is obtained by approximating a basis polynomial of the form

$$y(x) = \sum_{j=0}^{k+1} \alpha_j \left(\frac{x - x_n}{h}\right)^j, \tag{2.3}$$

$$y'(x) = \frac{1}{h} \sum_{j=0}^{k+1} j\alpha_j \left(\frac{x - x_n}{h}\right)^{j-1}. \tag{2.4}$$

As the normalization of the coefficients in (2.2) takes place in the first derivative part, the basis polynomial in (2.4) is employed to derive the continuous method through interpolating $y(x)$ at point x_{n+p_j} and collocating $y'(x)$ at x_{n+p_i}, $i, j = 0(1)k$. That is

$$\begin{aligned} y(x_{n+p_j}) &= y_{n+p_j}; \quad j = 0, 1, 2, 3, \cdots, k, \\ y'(x_{n+p_{i-1}}) &= y'_{n+p_{i-1}}; \quad i = 0, 1, 2, 3, \cdots, k. \end{aligned} \tag{2.5}$$

This leads to a system of $k+1$ equation given in a compact form

$$\begin{pmatrix} 1 & p_0 & p_0^2 & p_0^3 & \cdots & p_0^k & p_0^{k+1} \\ 1 & p_1 & p_1^2 & p_1^3 & \cdots & p_1^k & p_1^{k+1} \\ 1 & p_2 & p_2^2 & p_2^3 & \cdots & p_2^k & p_2^{k+1} \\ 1 & p_3 & p_3^2 & p_3^3 & \cdots & p_3^k & p_3^{k+1} \\ \vdots & & & & & \vdots & \\ 1 & p_k & p_k^2 & p_k^3 & \cdots & p_k^k & p_k^{k+1} \\ 0 & 1 & 2p_{i-1} & 3p_{i-1}^2 & \cdots & kp_{i-1}^{k-1} & (k+1)p_{i-1}^k \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_k \\ \alpha_{k+1} \end{pmatrix} = \begin{pmatrix} y_n + p_0 \\ y_{n+p_1} \\ y_{n+p_2} \\ y_{n+p_3} \\ \vdots \\ y_{n+p_k} \\ hf_{n+p_{i-1}} \end{pmatrix} \tag{2.6}$$

where, $p_0 = 0$. This system determines α_j, $j = 0(1)k+1$, which are used to derive the continuous schemes

$$\sum_{j=0}^{k} \alpha_{i,j} y_{n+p_j} - h\alpha_{i,k+1} f_{n+p_{i-1}} = hy\prime(x), \quad i = 1, 2, \cdots, k. \tag{2.7}$$

Replacing $x = x_{n+p_i}$ for $i = 1, 2, \cdots, k$ in (2.7) gives the block hybrid trapezoidal-type methods (BHTMs). The new continuous schemes in (2.7) preserves the known Runge-Kutta method of self-starting property. The BHTMs in (2.7) yield blocks of solutions upon application, while conventional linear multistep methods produce a single solution. However, when discrete points, x_{n+p_j}, $j = 0(1)k$ are considered for the methods in (2.2), they reduce to the conventional trapezoidal rule of the second kind [26],

$$A_1 Y_{n+1} = A_0 Y_n + h\left(B_1 F_{n+1} + B_0 F_n\right) \tag{2.8}$$

where the coefficient matrices are defined as

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{2,k+1} & 1 & 0 & \ddots & 0 & 0 \\ 0 & \alpha_{3,k+1} & 1 & \ddots & \vdots & 0 \\ 0 & 0 & \alpha_{3,k+1} & \ddots & 0 & 0 \\ & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ddots & \alpha_{k,k+1} & 1 \end{pmatrix}_{N\times N}, \quad B_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha_{1,k+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{N\times N}, \tag{2.9}$$

$$A_1 = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,k} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \alpha_{k-1,1} & \alpha_{k-1,2} & \cdots & \alpha_{k-1,k} \\ \alpha_{k,1} & \alpha_{k,2} & \cdots & \alpha_{k,k} \end{pmatrix}_{N\times N}, \quad A_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha_{1,0} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{2,0} \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \\ & \vdots & & \ddots & 0 & \alpha_{k-1,0} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{k,0} \end{pmatrix}_{N\times N}, \tag{2.10}$$

and the block solution output and its derivative are

$$\begin{aligned} Y_{n+1} &= [y_{n+p_1}, y_{n+p_2}, \cdots, y_{n+p_k}]^T, \\ F_{n+1} &= [f_{n+p_1}, f_{n+p_2}, \cdots, f_{n+p_k}]^T, \\ Y_n &= \left[y_{n-p_{k-1}}, y_{n-p_{k-2}}, \cdots, y_{n-p_1}, y_n + p_0\right]^T, \\ F_n &= \left[f_{n-p_{k-1}}, f_{n-p_{k-2}}, \cdots, f_{n-p_1}, f_n + p_0\right]^T, \end{aligned} \tag{2.11}$$

respectively.

2.1 Derivation of one block two-point schemes

To fix idea, for $k = 2$ in (2.2), the hybrid points are $x_0, x_{\frac{1}{2}}, x_1$. Then the basis function of $y(x)$ and $y\prime(x)$ are

$$y(x) = \sum_{j=0}^{3} \alpha_j \left(\frac{x - x_n}{h}\right)^j, \tag{2.12}$$

$$y'(x) = \frac{1}{h}\sum_{j=0}^{3} j\alpha_j \left(\frac{x - x_n}{h}\right)^{j-1}, \tag{2.13}$$

respectively. Here, we interpolate at points x_n, $x_{n+\frac{1}{2}}$, x_{n+1} and collocate at $x_{n+p_{i-1}}$, $i = 1, 2$. The system of equation from (2.12) and (2.13) in the compact form is given as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \left(\frac{1}{2}\right)^2 & \left(\frac{1}{2}\right)^3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2p_{i-1} & 3\left(p_{i-1}\right)^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ hf_{n+p_{i-1}} \end{pmatrix}. \tag{2.14}$$

By solving the system (2.14) for the coefficients α_i, $i = 0(1)3$ simultaneous, results into

$$\begin{aligned} \alpha_0 &= y_n \\ \alpha_1 &= \frac{-1}{6p_{i-1}^2 - 6p_{i-1} + 1}\Big(-hf_{p_{i-1}+n} + 18p_{i-1}^2 y_n + 6p_{i-1}^2 y_{n+1} - 24p_{i-1}^2 y_{n+\frac{1}{2}} \\ &\qquad -14p_{i-1}y_n - 2p_{i-1}y_{n+1} + 16p_{i-1}y_{n+\frac{1}{2}}\Big) \\ \alpha_2 &= -\frac{3hf_{p_{i-1}+n} - 12p_{i-1}^2 y_n - 12p_{i-1}^2 y_{n+1} + 24p_{i-1}^2 y_{n+\frac{1}{2}} + 7y_n + y_{n+1} - 8y_{n+\frac{1}{2}}}{6p_{i-1}^2 - 6p_{i-1} + 1} \\ \alpha_3 &= -\frac{2\left(-hf_{p_{i-1}+n} + 4p_{i-1}y_n + 4p_{i-1}y_{n+1} - 8p_{i-1}y_{n+\frac{1}{2}} - 3y_n - y_{n+1} + 4y_{n+\frac{1}{2}}\right)}{6p_{i-1}^2 - 6p_{i-1} + 1} \end{aligned} \tag{2.15}$$

and by replacing α_i, $i = 0(1)3$ in (2.13) with (2.15) leads to

$$\begin{aligned} hy'(x) = &-\frac{-hf_{p_{i-1}+n} + 18p_{i-1}^2 y_n + 6p_{i-1}^2 y_{n+1} - 24p_{i-1}^2 y_{n+\frac{1}{2}} - 14p_{i-1}y_n - 2p_{i-1}y_{n+1} + 16p_{i-1}y_{n+\frac{1}{2}}}{6p_{i-1}^2 - 6p_{i-1} + 1} \\ &-\frac{2(\mathrm{x} - x_n)\left(3hf_{p_{i-1}+n} - 12p_{i-1}^2 y_n - 12p_{i-1}^2 y_{n+1} + 24p_{i-1}^2 y_{n+\frac{1}{2}} + 7y_n + y_{n+1} - 8y_{n+\frac{1}{2}}\right)}{h\left(6p_{i-1}^2 - 6p_{i-1} + 1\right)} \\ &-\frac{6(\mathrm{x} - x_n)^2\left(-hf_{p_{i-1}+n} + 4p_{i-1}y_n + 4p_{i-1}y_{n+1} - 8p_{i-1}y_{n+\frac{1}{2}} - 3y_n - y_{n+1} + 4y_{n+\frac{1}{2}}\right)}{h^2\left(6p_{i-1}^2 - 6p_{i-1} + 1\right)}. \end{aligned} \tag{2.16}$$

Evaluating (2.16) at point $x = x_{n+\frac{1}{2}}$, $i = 1$ and $x = x_{n+1}$, $i = 2$ results into the hybrid linear multistep methods

$$\begin{aligned} -\frac{y_n}{2} + \frac{2y_{n+\frac{1}{2}}}{5} + \frac{y_{n+1}}{10} &= \frac{h}{10}\left(f_n + 2f_{n+\frac{1}{2}}\right); \quad C_4 = \frac{1}{192} \\ \frac{y_n}{5} - \frac{4y_{n+\frac{1}{2}}}{5} + y_{n+1} &= \frac{h}{10}\left(f_{n+\frac{1}{2}} + 2f_{n+1}\right); \quad C_4 = \frac{-1}{96}. \end{aligned} \tag{2.17}$$

These constitute the building blocks of the hybrid trapezoidal-type methods

$$\begin{aligned} \begin{pmatrix} \frac{2}{5} & \frac{1}{10} \\ -\frac{4}{5} & 1 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{5} \end{pmatrix}\begin{pmatrix} y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} = h\begin{pmatrix} \frac{1}{5} & 0 \\ \frac{1}{10} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix} \\ +h\begin{pmatrix} 0 & \frac{1}{10} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}. \end{aligned} \tag{2.18}$$

2.2 The order, consistency and the error constant of the block schemes

Following Fatunla [18], we associate a linear difference operator $L[y(x);h]$ given by

$$L[y(x):h] = \sum_{j=0}^{k} \alpha_{i,j} y(x+p_j h) - h\left(\beta_{i,k+1} y'(x+p_{i-1}h) - y'(x+p_i h)\right). \tag{2.19}$$

Here, $y(x)$ is assumed to be a sufficiently differentiable function on the interval $[a,b] \in \mathbb{R}^2$. Equation (2.19) serves as the representation for the local truncation error (LTE) associated with the hybrid block scheme presented in (2.7), with $y(x)$ considered as the exact solution of (1.1). The transformation of (2.19) involves expanding the terms $y'(x+v_i h)$ and $y(x+jh)$ around t through Taylor series and then organizing the resulting expressions based on powers of h

$$L(y(x);h)] = \bar{C}_0 y(x) + \bar{C}_1 h y'(x) + \cdots + \bar{C}_p h^p y^p(x) + \cdots, \tag{2.20}$$

where

$$\begin{aligned}
\bar{C}_0 &= \sum_{j=0}^{k} \alpha_{i,j}, \\
\bar{C}_1 &= \sum_{j=0}^{k} j\alpha_{i,j} - \alpha_{i,k+1} - 1, \\
\bar{C}_2 &= \sum_{j=0}^{k} \frac{j^2}{(2)!}\alpha_{i,j} - (p_{i-1})\alpha_{i,k+1} - p_i \\
&\vdots \\
\bar{C}_q &= \frac{(p_i)^q}{q!} - \sum_{j=0}^{k} \frac{j^q}{(q)!}\alpha_{i,j} - \frac{j^{q-1}}{(q-1)!}\left((p_{i-1})\alpha_{i,k+1} + p_i\right), \quad q \geq 1, \quad k \geq 1
\end{aligned} \tag{2.21}$$

with $\bar{C}_q = [C_{1,q}, C_{2,q}, \cdots, C_{k,q}]^T$.

The following definitions hold:

Definition 2.1. The hybrid scheme (2.7) is of order p, if

$$\bar{C}_j = 0, \quad j = 0(1)p, \quad \bar{C}_{p+1} \neq 0, \tag{2.22}$$

where $\bar{C}_{p+1} \neq 0$ is the error constant (EC) and its principal local truncation error (LTE) is

$$LTE = \bar{C}_{p+1} h^{p+1} y^{p+1}(x_n) + O(h^{p+2}) \tag{2.23}$$

from (2.20). The local truncation error of each constituent linear multistep methods in the schemes (2.8) is displayed below:

The LTE of the one-block 2-point method is as,

$$L\left[y(x_n;h)\right]=\begin{cases}\frac{h^4y^{(4)}(t)}{192}+O\left(h^5\right)\\ \frac{-h^4y^{(4)}(t)}{96}+O\left(h^5\right)\end{cases}. \tag{2.24}$$

Therefore, for the BHTMs with $k=2$, we have

$$\bar{C}_0=\bar{C}_1=\cdots=\bar{C}_3=0 \tag{2.25}$$

and

$$\bar{C}_4=min\left(\frac{1}{192},\frac{-1}{96}\right)^T \tag{2.26}$$

showing that the BHTMs with $k=2$, posses minimum order $p=3$.

The LTE of the one-block 3-point method is as,

$$L\left[y(x_n;h)\right]=\begin{cases}-\frac{h^5y^{(5)}(t)}{4860}+O\left(h^6\right)\\ \frac{h^5y^{(5)}(t)}{4860}+O\left(h^6\right)\\ -\frac{h^5y^{(5)}(t)}{1620}+O\left(h^6\right)\end{cases}. \tag{2.27}$$

Therefore, for the BHTMs with $k=3$, we have

$$\bar{C}_0=\bar{C}_1=\cdots=\bar{C}_4=0 \tag{2.28}$$

and

$$\bar{C}_5=min\left(-\frac{1}{4860},\frac{1}{4860},\frac{-1}{1620}\right)^T \tag{2.29}$$

showing that the BHTMs with $k=3$, posses order $p=4$.

The LTE of the one-block 4-points as,

$$L\left[y(x_n;h)\right]=\begin{cases}\frac{h^6y^{(6)}(t)}{122880}+O\left(h^7\right)\\ -\frac{h^6y^{(6)}(t)}{184320}+O\left(h^7\right)\\ \frac{h^6y^{(6)}(t)}{122880}+O\left(h^7\right)\\ -\frac{h^6y^{(6)}(t)}{30720}+O\left(h^7\right)\end{cases}. \tag{2.30}$$

Therefore, for the BHTMs with $k=4$, we have

$$\bar{C}_0=\bar{C}_1=\cdots=\bar{C}_6=0 \tag{2.31}$$

and

$$\bar{C}_6 = min\left(\frac{1}{122880}, \frac{1}{184320}, \frac{1}{122880}, \frac{1}{30720}\right)^T \tag{2.32}$$

showing that the BHTMs with $k = 4$, posses order $p = 5$.

2.3 Zero stability of the block hybrid Trapezoidal-type methods

Given a one block method in (2.8), the scheme is said to be zero-stable if the

$$Det[A_1 R - A_0] = R^k - R^{k-1} = 0 \tag{2.33}$$

which is the first characteristics polynomial contain roots $|R| < 1$ and one root on the unit circle. We shall consider the case of two-point.

Zero stability for two-point schemes

For $k = 2$ and $h = 0$ in the block schemes (2.8), we have

$$\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{4}{5} & 1 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{5} \end{pmatrix}\begin{pmatrix} y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.34}$$

then the first characteristics stability polynomial is given as

$$\rho(R) = det\left[\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{4}{5} & 1 \end{pmatrix} R - \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{5} \end{pmatrix}\right] \tag{2.35}$$

$$\rho(R) = 12R^2 - 12R = 0. \tag{2.36}$$

The roots are $R = 0$ and $R = 1$.

3 Stability of the Block Schemes

In this section, the one block schemes in (2.8) shall be used to determine the stability properties of the methods in (2.2). The stability analysis is obtained by considering the Dahlquist test equation

$$y' = \lambda y, \quad Re(\lambda) < 0. \tag{3.1}$$

The application of the schemes

$$A_1 Y_{n+1} = A_0 Y_n + h\left(B_1 F_{n+1} + B_0 F_n\right), \tag{3.2}$$

generate the discrete solution of the form

$$Y_{n+1} = M(z)Y_n, \quad z = \lambda h, \tag{3.3}$$

where $M(z) = (A - zB)^{-1}(A_0 + zB_0)$ is the amplification matrix. The behaviour of the numerical solution Y_{m+1} will depend on the eigenvalue of $M(z)$. That is, the stability matrix $M(z)$ has eigenvalues $\{0, 0, \cdots, 0, H(z)\}$, where $H(z)$ is the dominant eigenvalue. The region of absolute stability $H(z)$ for the newly derived method is

$$\mathbb{H}(z) = \{z \in \mathbb{C} :| \mathrm{p}(H(z)) < 1\}, \tag{3.4}$$

and its profile is shown in Fig. 1. The methods are A-stable because the left half complex plane is contained in $\mathbb{H}(z)$. For example the spectral radius for the block method of order $p \leq 15$ is presented as follows:

$$H(z) = \frac{z^2 + 6z + 12}{z^2 - 6z + 12} \tag{3.5}$$

$$H(z) = -\frac{z^3 + 11z^2 + 54z + 108}{z^3 - 11z^2 + 54z - 108} \tag{3.6}$$

$$H(z) = \frac{3z^4 + 50z^3 + 420z^2 + 1920z + 3840}{3z^4 - 50z^3 + 420z^2 - 1920z + 3840} \tag{3.7}$$

$$H(z) = -\frac{12z^5 + 274z^4 + 3375z^3 + 25500z^2 + 112500z + 225000}{12z^5 - 274z^4 + 3375z^3 - 25500z^2 + 112500z - 225000} \tag{3.8}$$

$$H(z) = \frac{5z^6 + 147z^5 + 2436z^4 + 26460z^3 + 189000z^2 + 816480z + 1632960}{5z^6 - 147z^5 + 2436z^4 - 26460z^3 + 189000z^2 - 816480z + 1632960} \tag{3.9}$$

$$\begin{aligned} H(z) = -\Big(&30z^7 + 1089z^6 + 22981z^5 + 331681z^4 + 3361400z^3 + 23193660z^2 \\ &+98825160z + 197650320\Big)\Big/\Big(30z^7 - 1089z^6 + 22981z^5 - 331681z^4 + 3361400z^3 \\ &-23193660z^2 + 98825160z - 197650320\Big) \end{aligned} \tag{3.10}$$

$$\begin{aligned} H(z) = -\Big(&560z^9 + 28516z^8 + 879525z^7 + 19539360z^6 + 327229875z^5 \\ &+4151341530z^4 + 39060913500z^3 + 258918055200z^2 + 1084777369200z \\ &+2169554738400\Big)\Big/\Big(560z^9 - 28516z^8 + 879525z^7 - 19539360z^6 \\ &+327229875z^5 - 4151341530z^4 + 39060913500z^3 - 258918055200z^2 \\ &+1084777369200z - 2169554738400\Big) \end{aligned} \tag{3.11}$$

$$H(z) = \Big(756z^{10} + 44286z^9 + 1594197z^8 + 42047500z^7 + 854232500z^6 + 13530825000z^5 + 165661650000z^4 + 1524600000000z^3 + 9979200000000z^2 + 41580000000000z + 83160000000000\Big) \Big/ \Big(756z^{10} - 44286z^9 + 1594197z^8 - 42047500z^7 + 854232500z^6 - 13530825000z^5 + 165661650000z^4 - 1524600000000z^3 + 9979200000000z^2 - 41580000000000z + 83160000000000\Big) \tag{3.12}$$

$$H(z) = -\Big(75600z^{11} + 5022660z^{10} + 207512217z^9 + 6368113598z^8 + 153050791575z^7 + 2929561979025z^6 + 44602042113090z^5 + 531885183734520z^4 + 4813042826869200z^3 + 31195647951930000z^2 + 129417373789149600z + 258834747578299200\Big) \Big/ \Big(75600z^{11} - 5022660z^{10} + 207512217z^9 - 6368113598z^8 + 153050791575z^7 - 2929561979025z^6 + 44602042113090z^5 - 531885183734520z^4 + 4813042826869200z^3 - 31195647951930000z^2 + 129417373789149600z - 258834747578299200\Big) \tag{3.13}$$

$$H(z) = \Big(5775z^{12} + 430105z^{11} + 20120412z^{10} + 707007444z^9 + 19716205080z^8 + 445111524000z^7 + 8163027512640z^6 + 120649773404160z^5 + 1409867251752960z^4 + 12589541572608000z^3 + 80960436574617600z^2 + 334688120576409600z + 669376241152819200\Big) \Big/ \Big(5775z^{12} - 430105z^{11} + 20120412z^{10} - 707007444z^9 + 19716205080z^8 - 445111524000z^7 + 8163027512640z^6 - 120649773404160z^5 + 1409867251752960z^4 - 12589541572608000z^3 + 80960436574617600z^2 - 334688120576409600z + 669376241152819200\Big) \tag{3.14}$$

$$
\begin{aligned}
H(z) = -\Big(&9979200z^{13} + 825114960z^{12} + 43219665216z^{11} + 1716512136612z^{10} \\
&+54692687879830z^{9} + 1429243329704190z^{8} + 30835523891217045z^{7} \\
&+547508021091911880z^{6} + 7907729795188783560z^{5} + 90929633007908494800z^{4} \\
&+803304354604150198800z^{3} + 5132801850197946724800z^{2} \\
&+21157158845937877963200z + 42314317691875755926400\Big)\Big/\Big(9979200z^{13} \\
&-825114960z^{12} + 43219665216z^{11} - 1716512136612z^{10} \\
&+54692687879830z^{9} - 1429243329704190z^{8} + 30835523891217045z^{7} \\
&-547508021091911880z^{6} + 7907729795188783560z^{5} - 90929633007908494800z^{4} \\
&+803304354604150198800z^{3} - 5132801850197946724800z^{2} \\
&+21157158845937877963200z - 42314317691875755926400\Big)
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
H(z) = \Big(&579150z^{14} + 52727985z^{13} + 3063724983z^{12} + 136057801425z^{11} \\
&+4891474173950z^{10} + 145766510990100z^{9} + 3632662356763440z^{8} \\
&+75712885241994000z^{7} + 1310700124736402400z^{6} + 18584453396068560000z^{5} \\
&+210916968363945043200z^{4} + 1846928213558588160000z^{3} \\
&+11738254868394582528000z^{2} + 48266390647664437248000z \\
&+96532781295328874496000\Big)\Big/\Big(579150z^{14} - 52727985z^{13} + 3063724983z^{12} \\
&-136057801425z^{11} + 4891474173950z^{10} - 145766510990100z^{9} \\
&+3632662356763440z^{8} - 75712885241994000z^{7} + 1310700124736402400z^{6} \\
&-18584453396068560000z^{5} + 210916968363945043200z^{4} \\
&-1846928213558588160000z^{3} + 11738254868394582528000z^{2} \\
&-48266390647664437248000z + 96532781295328874496000\Big)
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
H(z) = -\Big(&336336z^{15} + 33481196z^{14} + 2140908894z^{13} + 105354077163z^{12} \\
&+4229395852500z^{11} + 141985140058125z^{10} + 4028109907696875z^{9} \\
&+96799532676721875z^{8} + 1963290266437500000z^{7} + 33300510196078125000z^{6} \\
&+465087989303437500000z^{5} + 5221281801517031250000z^{4} \\
&+45383898872812500000000z^{3} + 287140437099140625000000z^{2} \\
&+1178235836121093750000000z + 2356471672242187500000000\Big) \Big/ \Big(336336z^{15} \\
&-33481196z^{14} + 2140908894z^{13} - 105354077163z^{12} + 4229395852500z^{11} \\
&-141985140058125z^{10} + 4028109907696875z^{9} - 96799532676721875z^{8} \\
&+1963290266437500000z^{7} - 33300510196078125000z^{6} + 465087989303437500000z^{5} \\
&-5221281801517031250000z^{4} + 45383898872812500000000z^{3} \\
&-287140437099140625000000z^{2} + 1178235836121093750000000z \\
&-2356471672242187500000000\Big)
\end{aligned} \tag{3.17}
$$

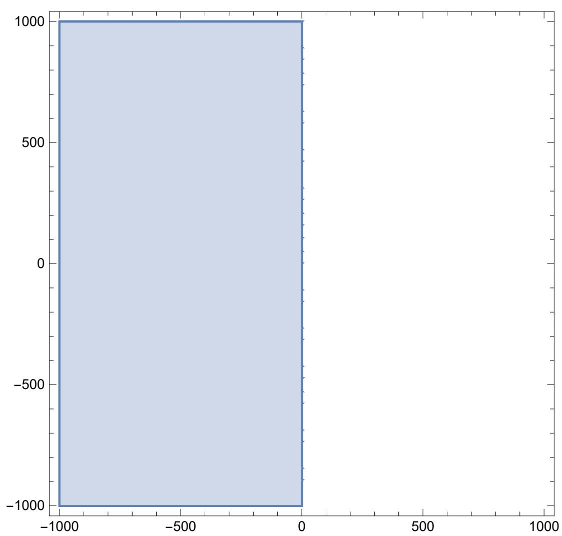

Figure 1: Region of absolute stability for the block hyrid method (2.8) for $k \leq 15$.

4 Numerical Experiments

The scheme is implemented more efficiently as block numerical integrators in (2.8). The procedure for block scheme (2.8) for the case of $k = 4$

$$\begin{pmatrix} -\frac{2}{3} & -3 & \frac{2}{3} & -\frac{1}{12} \\ \frac{44}{9} & -4 & -\frac{4}{3} & \frac{1}{9} \\ 2 & 6 & -\frac{22}{3} & -\frac{1}{2} \\ -\frac{8}{3} & 12 & \frac{8}{3} & -\frac{37}{3} \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{37}{12} \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{3}{4}} \\ y_{n} \end{pmatrix}$$
$$= h \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{3}{4}} \\ f_{n} \end{pmatrix} + h \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} \tag{4.1}$$

is given according to the following sequences.

Given the partition

$$\varpi_n : a = x_0 < x_1 < \cdots < x_u < x_{u+1} < \cdots < x_N, \quad h = x_{u+1} - x_u, \quad u = 0, 1, \cdots, N-1.$$

Stage 1: Fixed N for $k = 4$, $h = \frac{(b-a)}{N}$ the number of block $\Gamma = \frac{N}{4}$. Using (2.8), $n = 0$, $u = 0$, the solution value of $\left(y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_1\right)^T$ are simultaneously obtained over the sub-interval $[x_0, x_4]$ since y_0 is known from (2.1).

Stage 2: $n = 1$, $u = 4$, $\left(y_{\frac{5}{4}}, y_{\frac{3}{2}}, y_{\frac{7}{4}}, y_2\right)^T$ are similarly obtained over the sub-interval $[x_4, x_8]$ since y_1 is known from the previous block.

Stage 3: the iteration process is continued for $u = 8, \cdots, N-4$ and $n = 2, \cdots, \Gamma$ to get approximate solution to (2.1) on sub-intervals $[x_8, x_{12}] \cdots [x_{N-4}, x_N = b]$. Here, the subintervals do not over-lap. For linear problems, the code employs Gaussian elimination, whereas for nonlinear problems, Newton's method is utilized. In these cases, the block solution $Y_{n+1} = Y_{n+1}^{[q]}$, in (2.8) is determined iteratively from,

$$Y_{n+1}^{[i+1]} = Y_{n+1}^{[i]} - \left(\frac{\partial F(Y_{n+1}^{[i]})}{\partial Y_{n+1}}\right)^{-1} F(Y_{n+1}^{[i]}); \qquad i = 0(1)q \quad q > 1, \tag{4.2}$$

where

$$\frac{\partial F(Y_{n+1})}{\partial Y_{n+1}} = \begin{pmatrix} \frac{\partial f_{n+p_1}}{\partial y_{n+p_1}} & \frac{\partial f_{n+p_1}}{\partial y_{n+p_2}} & \cdots & \frac{\partial f_{n+p_1}}{\partial y_{n+p_k}} \\ \frac{\partial f_{n+p_2}}{\partial y_{n+p_1}} & \frac{\delta f_{n+p_2}}{\partial y_{n+p_2}} & \cdots & \frac{\partial f_{n+p_2}}{\partial y_{n+p_k}} \\ \vdots & & & \\ \frac{\partial f_{n+p_k}}{\partial y_{n+p_1}} & \frac{\partial f_{n+p_k}}{\partial y_{n+p_2}} & \cdots & \frac{\partial f_{n+p_k}}{\partial y_{n+p_k}} \end{pmatrix} \tag{4.3}$$

and

$$F(Y_{n+1}) = A_1 Y_{n+1} + A_0 Y_n - h\left(B_0 F_n + B_1 F_{n+1}\right) = 0. \tag{4.4}$$

In the Newton's scheme, the criteria for terminating non-linear problems which does not have theoretical solution is $\| Y_{n+1}^{[i+1]} - Y_{n+1}^{[i]} \| < Tol$, where Tol is the accuracy tolerance of the approximations, defined by the user.

4.1 Test problems

Problem 1: Considering the initial value problems (IVPs) presented in the works of Fotta *et al.* [19] and Rufai *et al.* [20]:

$$y\prime = -10(y-1)^2, \quad y(0) = 2, \quad x \in [0, 0.1]. \tag{4.5}$$

The analytical solution is $y = 1 + \frac{1}{1+10x}$. Employing the block hybrid schemes (2.8) with orders $p = 4$ (BHTMs3) and $p = 5$ (BHTMs4) on problem 1, the errors ($| y - y(x) |$) are calculated for various intervals $0 < x \leq 0.1$ and presented in Table 1. The numerical results and comparisons in Table 1 indicate that BHTMs4 outperform the methods with $p = 5$ in Fotta *et al.* [19], and the methods with $p = 6$ in Rufai *et al.* [20].

Problem 2:

$$\begin{aligned} y_1'(x) &= -2000y_1 + 1000y_2 + 1 \qquad y_1(0) = 0 \\ y_2'(x) &= y_1 - y_2 \qquad\qquad\qquad y_2(0)) = 0. \end{aligned} \tag{4.6}$$

The second problem exhibits stiffness, characterized by a stiffness ratio $S = 4001$, and the corresponding theoretical solution is given by

$$\begin{aligned} y_1(x) &= 4.97 \times 10^{-4} e^{-2000.5x} - 5.034 \times 10^{-4} e^{-0.5x} + 0.001 \\ y_2(x) &= -2.5 \times 10^{-7} e^{-2000.5x} - 1.007 \times 10^{-3} e^{-0.5x} + 0.001. \end{aligned} \tag{4.7}$$

The BHTMs4 scheme is compared with methods outlined in Ismail and Ibrahim [21], CB_2DF in [22], and ECB_2DF in [23] at a uniform order $p = 5$, as summarized in Table 2. The analysis of Table 2 reveals

Table 1: Numerical results for problem 1 , *error* $y = \mid y - y(x) \mid$, $h = 0.01$

T	BHTMs3 *error y*	BHTMs4 *error y*	Fotta *et al.*,(2015) *error y*	Rufai *et al.*,(2016) *error y*
0.01	1.227×10^{-7}	2.069×10^{-10}	2.829×10^{-7}	1.558×10^{-6}
0.02	1.769×10^{-7}	2.751×10^{-10}	4.045×10^{-7}	2.399×10^{-6}
0.03	1.946×10^{-7}	2.788×10^{-10}	4.472×10^{-7}	2.830×10^{-6}
0.04	1.973×10^{-7}	2.750×10^{-10}	4.509×10^{-7}	3.020×10^{-6}
0.05	1.905×10^{-7}	2.549×10^{-10}	4.356×10^{-7}	3.069×10^{-6}
0.06	1.801×10^{-7}	2.343×10^{-10}	4.117×10^{-7}	3.034×10^{-6}
0.07	1.677×10^{-7}	2.129×10^{-10}	3.846×10^{-7}	2.951×10^{-6}
0.08	1.562×10^{-7}	1.950×10^{-10}	3.572×10^{-7}	2.840×10^{-6}
0.09	1.452×10^{-7}	1.762×10^{-10}	3.307×10^{-7}	2.717×10^{-6}
0.10	2.331×10^{-7}	1.611×10^{-10}	3.058×10^{-7}	2.588×10^{-6}

Table 2: Comparison of numerical schemes for problem 2

Method	$(\max \mid y_{1,h} - y_1(x) \mid)$	$(\max \mid y_{2,h} - y_2(x) \mid)$
BHTMs4	1.025×10^{-7}	1.748×10^{-7}
Ismail-Ibrahim	3.649×10^{-7}	7.670×10^{-7}
CB_2DF5	2.328×10^{-7}	5.021×10^{-7}
ECB_2DF5	2.328×10^{-7}	5.021×10^{-7}
BHTMs4	4.285×10^{-9}	4.682×10^{-9}
Ismail-Ibrahim	2.454×10^{-7}	4.942×10^{-7}
CB_2DF5	1.700×10^{-8}	3.705×10^{-8}
ECB_2DF5	1.700×10^{-8}	3.705×10^{-8}

that the BHTMs4 scheme exhibits superior accuracy compared to the methods proposed by Ismail and Ibrahim [21], Akinfenwa *et al.* [22], and Akinfenwa and Jator [23] at the time points $X = 5$ and 10.

Problem 3: Consider the linear problem in [26],

$$y'(x) = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y(x), \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \tag{4.8}$$

with solution,

$$y(x) = \frac{1}{2} \begin{pmatrix} e^{-2x} + e^{-40x}\left(cos(40x) + sin(40x)\right) \\ e^{-2x} - e^{-40x}\left(cos(40x) + sin(40x)\right) \\ 2e^{-40x}\left(cos(40x) - sin(40x)\right) \end{pmatrix}.$$

The system of ordinary differential equations (ODEs) is stiff, with a stiffness ratio of $S = 28.5$. Solving the problem outlined in (4.8), the step lengths $h = \{0.05, 0.025, 0.0125, 0.00625, 0.003125\}$ are utilized over the interval $0 < x \leq 1$ employing BHTMs2, BHTMs3, BHTMs4, BHTMs5. The maximum relative error $\max_{1<i<3} \mid y_i(x) - y_{i,h} \mid /(1+ \mid y_{i,h} \mid)$ with respect to the step length is presented in Table 3. Analysis of Table 3 shows that the new schemes at order 5 outperform the GSDLMMEs3 of order $p = 5$ in [24], the variable-step boundary value methods based on the reverse Adams method (Amodio) of order $p = 6$ in [25], and GAMs of order 5 in [26]. The exact solution at point x for problem 3 is denoted by $y_i(x)$.

Table 3: Numerical solution of problem 3 on interval $0 < x \leq 1$

h	BHTMs2 $p = 3$	BHTMs3 $p = 4$	BHTMs4 $p = 5$	GAMs3 $p = 3$	GAMs5 $p = 5$
0.05	3.05×10^{-1}	1.87×10^{-1}	1.11×10^{-1}	4.21×10^{-1}	2.24×10^{-1}
0.025	3.12×10^{-2}	2.00×10^{-2}	1.30×10^{-2}	1.00×10^{-1}	4.41×10^{-2}
0.0125	3.45×10^{-3}	1.98×10^{-3}	1.30×10^{-3}	2.85×10^{-3}	6.49×10^{-3}
0.00625	3.20×10^{-4}	1.57×10^{-4}	3.77×10^{-5}	7.09×10^{-3}	8.85×10^{-4}
0.003125	2.45×10^{-4}	7.43×10^{-6}	7.52×10^{-7}	1.77×10^{-3}	9.88×10^{-5}

Table 4: Numerical solution of problem 3 on interval $0 < x \leq 1$ (continuation of Table 3)

h	GSDLMMEs3 $p = 5$	Amodio $p = 6$
0.05	3.00×10^{-2}	5.70×10^{-2}
0.025	3.55×10^{-3}	8.70×10^{-3}
0.0125	2.26×10^{-4}	4.90×10^{-4}
0.00625	5.86×10^{-6}	1.20×10^{-5}
0.003125	1.14×10^{-7}	2.20×10^{-7}

Problem 4: A linear problem solved by

$$y'(x) = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} y(x), \quad y(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{4.9}$$

The application of BHTMs4 to solve problem 4 yields the maximum error ($\max \mid y_{i,h} - y_i(x) \mid$) within the interval $0 < x \leq 10$. Table 5 reveals that the BHTMs4 scheme, with an order of $p = 5$, has a slight advantage over the methods (GSDLMMEs3) presented in [24], second derivative methods in [4], and [10] of order $p = 5$ with fixed step sizes $h = \{0.01, 0.08\}$.

Problem 5: Consider the non-linear system,

$$\begin{matrix} y_1' = -1002y_1 + 1000y_2^2, & y_1(0) = 1 \\ y_2' = y_1 - y_2(1 + y_2), & y_2(0) = 1 \end{matrix}, \quad y(x) = \begin{pmatrix} e^{-2x} \\ e^{-x} \end{pmatrix} \tag{4.10}$$

in Akinfenwa *et al.* [22], and Wu and Xia [27]. The problem in 5 was integrated using the schemes BHTMs5 in the interval $0 < x \leq 10$. Table 6 shows that the BHTMs5 outperform the compared methods (CB_2DF6) in [22] and methods in Wu and Xia [27] at time $X = 10$.

Table 5: Comparison of numerical scheme for problem 4

Method	h	$(\max \mid y_{i,h} - y_i(x) \mid)$
BHTMs4	0.08	0.99×10^{-8}
GSDLMMEs3	0.08	0.69×10^{-7}
[4]		0.44×10^{-6}
[10]	0.08	0.34×10^{-6}
[10]	0.01	0.11×10^{-10}
BHTMs4	0.01	0.28×10^{-11}
GSDLMMEs3	0.01	0.55×10^{-11}
ODE15s		0.63×10^{-6}

Table 6: Comparison of results for problem 5,
$Y = Max \mid y_i - y_i(x) \mid$

Methods	order p	x	h	Y
BHTMs5	6	1	0.02	1.93×10^{-13}
		10	0.02	1.24×10^{-19}
CB_2DF6	6	1	0.02	1.25×10^{-12}
		10	0.02	1.35×10^{-15}
Wu and Xia	6	1	0.002	2.56×10^{-7}
		10	0.002	6.09×10^{-12}

5 Conclusion

This article introduced a Block Hybrid Trapezoidal-type Methods (BHTMs) for solving ODEs. The theoretical framework of the BHTMs, including their convergence and stability properties, is thoroughly examined.The derived BHTMs exhibit A-stability for order $p \leq 15$, highlighting their robustness in practical applications. Moreover, the numerical results in Tables 1–6 indicate an enhancement in accuracy with the BHTMs compared to certain existing methods in the literature.

Acknowledgements

This is to gracefully acknowledge with gratitude the immense contribution of the anonymous reviewer which has greatly improved the quality of this work.

Conflict of interest

The authors have no conflicts of interest to declare.

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