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Optimal Investment and Debt Management Strategies in the Presence of Inflation Risks for a DC Pension Scheme

C. B. Ibe[1,*] and O. E. Daudu[2]

[1] Department of Mathematics, Faculty of Physical Sciences, University of Benin, P. M. B. 1154, Benin City, Edo State, Nigeria
e-mail: chiedozie.ibe@uniben.edu

[2] Department of Mathematics, Faculty of Physical Sciences, University of Benin, P. M. B. 1154, Benin City, Edo State, Nigeria
e-mail: oluwagbemigadaudu@gmail.com

Abstract

This paper derives the optimal debt ratio, portfolio strategies and consumption rate for a Pension Fund Administrator (PFA) in a defined contributory (DC) pension scheme. Four asset classes were considered which include riskless asset, inflation-linked bonds, stocks and housing. Five background risks which include inflation, stock, housing, income growth rate and salary risks are considered. The plan member contributes a proportion of his or her stochastic salary into the scheme. The PFA make further effort to borrow money to finance her investment to ensure maximum returns. The contribution of the plan member in addition to borrowed capital is invested in financial and non-financial (housing) assets. In this paper, the real wealth process of the PFA is considered. The problem is formulated as a bi-objective stochastic control problem. The resulting Hamilton-Jacobi-Bellman (HJB) equation was solved using dynamic programming approach for stochastic process. This paper aims at (i) maximize the total expected discounted utility of consumption and debt of the PFA in an infinite time horizon, (ii) determine the optimal debt ratio and optimal consumption plan of the PFA, (iii) determine the optimal investment strategies for a PFA who invest in an economy that is exposed to five background risks. Using dynamic programming approach, we derive the optimal debt ratio, optimal consumption plan, optimal investment in inflation-linked bonds, stocks and housing for a PFA that chooses the power utility function. We found that the optimal debt ratio depends directly on the optimal real wealth. We also found that the optimal investment in stocks, housing and inflation-linked bonds depend on inflation, stock, salary, housing and income growth rate risks. Numerical implementation of the models using empirical data are presented.

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*Corresponding author

1 Introduction

This paper focuses on solving the problem of a PFA under a defined contribution pension scheme that desire to increase her portfolio performance so as to attract more pension plan members (PPMs). In Nigeria, it is required that a Retirement Savings Account (RSA) is opened by the PFA for employees. The PFA then invest and manage the funds in the individual RSA in fixed income securities as outlined in the Nigerian Pension Reform Act NPRA (2004) and (2014) and other instruments as prescribed by the pension commission from time to time. The benefits that accrue to PPMs in a DC pension scheme, depends greatly on the portfolio choice of the PFA. Hence, an efficient investment strategy is key during the accumulation phase of the pension plan. In other words, a PPM needs to choose a PFA with the required competence to produce positive returns on his or her investment. This is because the benefits to PPMs at retirement depends on how well the PFA is able to manage their contributions. Clearly, in a DC pension scheme, a PFA with a good investment returns will be attractive to a PPM. In a bid to increase their portfolio value and make them more attractive to the PPM, we assume that the PFA borrows capital in addition to the contributions of the plan member to invest in both the financial and fixed assets markets. We note that the performance of an investment portfolio is affected by debt and the debt ratio gives an insight to the financial health of an organisation. In this paper, we look at debt from the PPM's point of view. Hence, we see the contributions by PPMs as debt that will be paid at retirement by the PFA.

The investment horizon of DC pension scheme is usually over a long period of time and it is therefore necessary for the PFA to consider the impact of inflation on its investment portfolio. Hence, we consider real wealth as a guide to investors for making the right investment decision. Battacchio and Menoncin (2004) studied the asset allocation problem for a DC pension in a complete financial market with stochastic investment opportunities in the presence of income and inflation risk and a one-factor stochastic inflation model was considered by Korn and Kruse (2004). Zhang and Ewald (2009) investigated the optimal investment problem for a DC pension scheme. The objective was to maximize the expected utility of the terminal value of the pension fund and they used the Martingale method to solve the resulting optimization problem. Zhang et al. (2009) and Maurer et al. (2008) assumed an inflation rate dynamics that follow an Ornstein-Uhlenbeck process. Nkeki (2017) considered a three factor inflation model with jumps and Want et al. (2020) considered the impact of inflation risk on robust optimal portfolios for DC pension fund in the presence of model ambiguity. Mi et al. (2023) considered an optimal investment problem for a DC pension fund under tail value at Risk and portfolio insurance constraints.

The literature on asset allocation and optimal investment is quite rich. Merton (1969) was the first to study the optimal portfolio selection problem within the framework of continuou-time models. Using stochastic optimal control methodology, he obtained the optimal portfolio strategy that maximize expected utility of terminal wealth. A more rigorous analysis of the Merton problem was provided by Karatzas et al. (1986) and Karatzas et al. (1991) provided a generalization of the model for an incomplete market using the Martingale method. Koo (1999) considered the optimal portfolio choice problem for an investor with liquidity problems and unhedge income risk. He found that exposure to unhedgable income risk and liquidity constraint, have a negative impact on the fraction of wealth the investor is willing to invest in risky assets and Gozzi et. al (2008) formulation and studied a continuous time stochastic model of optimal asset allocation for a DC pension fund with a minimum guarantee. Gaivoronski et al. (2005) investigated different portfolio selection method when the performance of the portfolio is appraised relative to a benchmark. They proposed different portfolio selection algorithms based on risk preferences. They also investigated the issue of dynamic portfolio rebalancing using various risk measures like variance and Value-at-risk. Altarovici et al. (2015) studied the optimal investment and consumption problem in the presence of transaction cost. An investor with CRRA that trades a safe and several risky assets with constant investment opportunities was considered. They assumed small transaction cost for each trade regardless of the size and obtained the optimal investment policy. There exist a myriad of literature on portfolio selection problem. For more details on optimal portfolio selection models see Bo et al. (2013), Castaneda-Leyva and Hernandez (2005), Jiao et al. (2013), Campbell and Viceira (1999), Li and Ng (2000), Nkeki and Nwozo (2013), Giacinto et al. (2011) etc.

Debt management has been attracting a lot of attention from researchers in recent times. Stein (2003) derived an optimal sustainable external debt for an economy by using stochastic optimal control. He suggested the use of the deviation between actual and optimal debt as a signal to a crisis. Krouglov (2013) developed and applied mathematical model to solving an economic problem in the presence of financial crisis. Jin (2014) studied the optimal debt ratio and consumption plan for an investor during financial crisis in a stochastic setting. The impact of labor market condition was also studied. He assumed that the production rate function of the investor is stochastic and being influenced by the government policy, employment and unanticipated risks. Lui and Jin (2014) proposed a stochastic optimal control debt management model for the estimation of optimal debt ratio for both the public and private sectors at two different market regimes. The decision makers aimed at maximizing the utility of terminal wealth by choosing the optimal debt ratio. They further considered the hidden Markov chain technique in order to estimate the actual situation of the strategy. Jin et al. (2015) considered the surplus process of an insurer which is assumed to follow a diffusion process. They derived the optimal debt ratio and dividend payment policies for an insurer by maximizing the expected discounted utility of dividend payment subject to the surplus process of the insurer. Nkeki (2018) considered a portfolio model to determine the optimal investment strategy, consumption plan and optimal debt ratio with backup security during a financial crisis and Zhao et al. (2018) obtained the optimal debt ratio, dividend payment and investment strategies for an insurance firm in the presence of unanticipated shocks. Their objective was to maximize the total expected discounted utility of dividend payment in a finite-time period.

The motivation for this paper is to study the investment strategy and debt ratio of the PFA in a stochastic environment. We assume that what accrues to the plan members is a function of the PFA portfolio performance and a debt which has to be paid. The contributions by the PPM and what is borrowed by the PFA is invested into a market that is divided into financial assets and fixed asset (housing) and modelled by Brownian motions. The investment portfolio of the PFA is a combination of financial and fixed assets. The resulting nominal wealth is the difference between the total investment portfolio and liability which include contribution by plan members and other components. Our interest is to obtain the optimal debt ratio, consumption plan and optimal portfolio strategies of the PFA in an investment environment with diffusion risks.

The highlights of this research work are as follows:

1. housing dynamics with depreciation parameter is considered.
2. five background risks are considered.
3. debt profile of a PFA is considered.
4. optimal investment in inflation-linked bonds, stocks and housing are obtained.
5. optimal debt ratio and optimal consumption plan for the investor are obtained.
6. empirical data were used to analyse the resulting models.

The remainder of the paper is structured as follows. In Section 2, we present the PFA's investment portfolio dynamics, the contribution process, income growth rate, debt process, nominal and real wealth process of the PFA. Section 3 presents the optimal controls, value function, optimal investments, optimal debt ratio and optimal consumption. Section 4 presents the empirical results of our models and we conclude the paper in Section 5.

2 Model Formulation

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ which satisfies the usual conditions i.e $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is right-continuous and $\mathbf{P}$ is complete real world probability measure, $\mathcal{F}_t$ is a filtration that represents information avaliable at time t and T is retirement date. We assume that all stochastic processes used below are well-defined on the given probability space and adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$.

2.1 The asset of the PFA

We begin by assuming that the pension fund administrator (PFA) manages his portfolio with the following assets: inflation-linked bonds $B(t, I(t))$, housing or real estate (fixed asset) $H(t)$ and riskless asset $A(t)$ at time t. The inflation-linked bonds is defined in terms of the inflation index $I(t)$ at time t as $B(t, I(t)) = e^{-\int_t^T \bar{r}(s)ds} I(t)$. We assume that $I(t)$ is correlated with inflation and housing price risks and has the dynamics

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_{1I}(t)dW_I(t) + \sigma_{2I}(t)dW_P(t),\ I(0) = I_0, \tag{1}$$

where $\mu_I(t) = r(t) - \bar{r}(t) + \sigma_{1I}(t)\theta_{1I}(t) + \sigma_{2I}(t)\theta_{2I}(t)$ represents the expected inflation index at time t, r(t) is instanteous interest rate and there is no default risk at time t, $\bar{r}(t)$ is the real interest rate at time t, $\sigma_{1I}(t) \in [\mathbb{R}^m \times [0, T]]$ represents the volatility of the price index with respect to inflation sources of risks, $\sigma_{2I}(t) \in [\mathbb{R}^q \times [0, T]]$ represents the volatility of the price index with respect to housing price sources of risks, $\theta_{1I}(t) \in [\mathbb{R}^m \times [0, T]]$ is the market price inflation risks, $\theta_{2I}(t) \in [\mathbb{R}^q \times [0, T]]$ is the market price of housing price risks, $W_I(t)$ is an m-dimensional Brownian motions that captures inflation sources of risks at time t and $W_P(t)$ a q-dimensional Brownian motions that captures sources of housing price risks at time t.

Re-writing (1) in compact form, we have

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW(t),\ I(0) = I_0, \tag{2}$$

where $\sigma_I(t) = (\sigma_{1I}(t), \sigma_{2I}(t))$, $W(t) = (W_I(t), W_P(t))'$ is an $m + q$-dimensional Brownian motions with respect to inflation risk and housing risk respectively at time, and the sign $'$ represent transpose.

We assume that the housing asset $H(t)$ is given by

$$H(t) = Q(t)P(t) \tag{3}$$

where $P(t) \in [\mathbb{R}^q \times [0, T]]$ is a column vector of housing asset prices at time t, $Q(t) \in [\mathbb{R}^q \times [0, T]]$ the row vector per unit size of housing at time t satisfying the dynamics $dQ(t) = -\zeta(t)Q(t)dt$, $\zeta(t)$ is the depreciation rate of housing per unit size at time t.

The housing price $P(t)$ is assumed to satisfy the following diffusion process

$$\frac{dP(t)}{P(t)} = \mu_P(t)dt + \sigma_P(t)dW(t), P(0) = p_0 \in \mathbb{R}_+, \tag{4}$$

where $\mu_P(t) \in [\mathbb{R}^q \times [0, T]]$ is the expected growth rate at time t, $\sigma_P(t) = (\sigma_{P_1}(t), \sigma_{P_2}(t))$ is the housing price volatility at time t such that $\sigma_{P_1}(t) \in [\mathbb{R}^{q\times m} \times [0, T]]$ is the volatility arising from inflation index sources of risks, $\sigma_{P_2}(t) \in [\mathbb{R}^{q\times q} \times [0, T]]$ is the volatility from housing prices source of risk at time t and

$\Sigma_H(t) = \sigma_P(t)\sigma'_P(t)$ is a positive definite and nonsingular matrix.

To obtain the dynamics of $H(t)$, we take the differential of bothsides of (3). It follows that

$$dH(t) = dQ(t)P(t) + Q(t)dP(t) \tag{5}$$

But $Q(t)dP(t)$ can be written as

$$\begin{aligned} \frac{Q(t)dP(t)}{P(t)}P(t) &= Q(t)P(t)\frac{dP(t)}{P(t)} \\ &= H(t)\frac{dP(t)}{P(t)}. \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} dH(t) &= H(t)\big(\mu_P(t)dt + \sigma_P(t)dW(t)\big) - \zeta(t)\mathbf{I}Q(t)P(t) \\ &= H(t)\big((\mu_P(t) - \zeta(t)\mathbf{I})dt + \sigma_P dW(t)\big), \end{aligned} \tag{7}$$

where $\mathbf{I} = (1, 1, 1...., 1)' \in \mathbb{R}^m$.

Therefore, the evolution of the assets of the PFA at time t are respectively given by the following equations:

$$\frac{dB(t, I(t))}{B(t, I(t))} = (r(t)\mathbf{I} + \sigma_B(t)\theta_B(t))dt + \sigma_B(t)dW(t), B(0) = b_0 \in \mathbb{R}_+. \tag{8}$$

$$dH(t) = H(t)\big((\mu_P(t) - \zeta(t)\mathbf{I})dt + \sigma_P dW(t)\big), H(0) = h_0 \in \mathbb{R}_+, \tag{9}$$

$$dA(t) = rA(t)dt, A(0) = A_0 \in \mathbb{R}, \tag{10}$$

where $\sigma_B(t) = (\sigma_{1B}(t), \sigma_{2B}(t))$ is the volatility tensor of inflation-linked bonds such that $\sigma_{1B}(t) \in [\mathbb{R}^{m\times m} \times [0, T]]$, $\sigma_{2B}(t) \in [\mathbb{R}^{m\times q} \times [0, T]]$ and $\Sigma_B(t) = \sigma_B(t)\sigma'_B(t)$ is a positive definite and nonsingular matrix, $\theta_B(t) = (\theta_{1I}(t), \theta_{2I}(t))'$ is the market price of risks at time t. From now on, we assume that $m = q$.

2.2 The asset value of the PFA

Let $K(t)$ be the total asset value of the PFA at time t such that $K_F(t)$ is the asset value in financial market at time t and $K_H(t)$ is the asset value in housing at time t. Hence, the total asset value of the PFA is defined as

$$K(t) = K_F(t) + K_H(t). \tag{11}$$

Let $\Delta_B(t) \in \mathbb{R}^m$ be the amount of funds invested in inflation-linked bonds, $\Delta_H(t) \in \mathbb{R}^q$ is the amount of funds invested in housing and $\Delta_0(t) = K(t) - \Delta_B(t)\mathbf{I} - \Delta_H(t)\mathbf{I}$ is the amount invested in riskless asset at time t.

Definition 1. *The wealth dynamics of assets in the financial market is defined as*

$$dK_F(t) = \Delta_0(t)\frac{dA(t)}{A(t)} + \Delta_B(t)\frac{dB(t,I(t))}{B(t,I(t))}. \tag{12}$$

Substituting (8) and (10) in (12), we obtain the following

$$\begin{aligned} dK_F(t) = &\big(r(t)K(t) + \Delta_B(t)\sigma_B(t)\theta_B(t) - \Delta_H(t)r(t)\mathbf{I}\big)dt \\ &+ \Delta_B(t)\sigma_B(t)dW(t),\ K_F(0) = K_{F0}. \end{aligned} \tag{13}$$

Definition 2. *The wealth dynamics of assets in housing is defined as*

$$dK_H(t) = K_H(t)\Delta_H(t)\frac{dH(t)}{H(t)}, K_H(0) = K_{H0} \in \mathbb{R}. \tag{14}$$

Using (9), we have that

$$\begin{aligned} &dK_H(t) = K_H(t)\Delta_H(t)\big(\mu_P(t) - \zeta(t)\mathbf{I})dt + K_H(t)\Delta_H(t)\sigma_P(t)dW(t), \\ &K_H(0) = K_{H0} \in \mathbb{R}. \end{aligned} \tag{15}$$

Proposition 1. *The dynamics of the total asset value of the PFA is*

$$\begin{aligned} &dK(t) = \big[r(t)K(t) - \Delta_H(t)\big(r(t)\mathbf{I} - (K(t) - K_F(t))(\mu_P(t) - \zeta(t)\mathbf{I}\big) \\ &+\Delta_B(t)\sigma_B(t)\theta_B(t)\big]dt + (\sigma_B(t)\Delta_B(t) + \Delta_H(t)\sigma_P(t)(K(t) - K_F(t)))dW(t). \end{aligned} \tag{16}$$

Proof. Taking the differential of both sides of (11), we have the following

$$dK(t) = dK_F(t) + dK_H(t). \tag{17}$$

We substitute (13) and (15) into (17), we have the following

$$\begin{aligned} dK(t) = &\big(r(t)K(t) + \Delta_B(t)\sigma_B(t)\theta_B(t) - \Delta_H(t)r(t)\mathbf{I}\big)dt \\ &+\Delta_B(t)\sigma_B(t)dW(t) + K_H(t)\Delta_H(t)\big(\mu_P(t) - \zeta(t)\mathbf{I})dt \\ &+K_H(t)\Delta_H(t)\sigma_P(t)dW(t). \end{aligned} \tag{18}$$

But $K_H(t) = K(t) - K_F(t)$, which implies that (18) becomes

$$\begin{aligned} &dK(t) = \big[r(t)K(t) - \Delta_H(t)\big(r(t)\mathbf{I} - (K(t) - K_F(t))(\mu_P(t) - \zeta(t)\mathbf{I}\big) \\ &+\Delta_B(t)\sigma_B(t)\theta_B(t)\big]dt + \sigma_B(t)\Delta_B(t) + \Delta_H(t)\sigma_P(t)(K(t) - K_F(t)))dW(t). \end{aligned} \tag{19}$$

The dynamics of the PFA's total asset value (19) is expressed in terms of $K_F(t)$. □

2.3 The contribution process, income growth rate and debt process of the PFA

The PPM is assumed to make a flow of contributions to the pension fund from his or her salary at time t. Let $Y(t)$ be the salary process of a PPM at time t which we assume is correlated with inflation index and housing prices, then $Y(t)$ satisfies the following dynamics

$$dY(t) = Y(t)\big(\beta(t)dt + \sigma_A dW(t) + \sigma_Y(t)dW_Y(t)\big), Y(0) = y_0 \in \mathbb{R}_+, \tag{20}$$

where $\beta(t) \in \mathbb{R}_+$ is the expected growth rate of the salary process of PPM at time t, $\sigma_A(t) = (\sigma_{Y_1}(t), \sigma_{Y_2}(t))$ is the volatilty tensor of the salary process of a PPM such that $\sigma_{Y_1}(t) \in [\mathbb{R}^m \times [0,T]]$ is the volatility arising from the uncertainity in inflation source of risks, $W_I(t)$ and $\sigma_{Y_2}(t) \in [\mathbb{R}^q \times [0,T]]$ is the volatility arising from the uncertatinity in housing price sources of risks, $W_P(t)$, $\sigma_Y(t) \in [\mathbb{R} \times [0,T]]$ is the volatility arising from the uncertatinity in salary sources of risks, $W_Y(t)$ and $W_Y(t)$ is a 1-dimensional Brownian motion that captures salary source of risk at time t.

Let $\hat{Y}(t)$ be the contribution received by the PFA from the PPM at time t. Suppose $\tilde{q}$ is the constant proportion of $Y(t)$ that is contributed by the PPM at time t, then $\hat{Y}(t) = Y(t)\tilde{q}$ is the amount contributed by a PPM at time t. The dynamics of $\hat{Y}(t)$ is given as

$$d\hat{Y}(t) = \hat{Y}(t)\big(\beta(t)dt + \sigma_A(t)dW(t) + \sigma_Y(t)dW_Y(t)\big), \hat{Y}(0) = \hat{y}_0 \in \mathbb{R}_+. \tag{21}$$

The aim of the PFA is to have a good number of PPMs as this helps with increasing their asset under management and investment diversification. We assume that the PFA does not depend only on PPM's contribution for portfolio expansion, but go extra mile to seek more fund to expand their investment. This is because a good portfolio profile and positive returns on investment will help attract PPMs. To this end, we assume that the PFA borrow a certain amount on which it pays the interest rate $r_L(t)$ at time t to finance investment in addition to pension contribution and this contribution is a liability on the part of the PFA to the PPM. We note that contribution by the PPM is held in trust by the PFA on behalf of the plan member and invested to generate income which is to be collected by the PPM at retirement.

Let $L(t)$ be the liability process at time t and is described as the difference between expenditure and income of the PFA at time t in addition to contribution by the PPMs. The expenditure component of $L(t)$ includes interest on debt and consumption flow $C(t)$. Given that $C(t)$ is the consumption flow of the PFA at time t, we have

$$dC(t) = c(t)X(t)dt, \tag{22}$$

where $c(t)$ measures the rate of consumption and $X(t)$ is the wealth process of the PFA at time t.

The income process $Z(t)$ of the PFA at time t is assumed to be affected by an expansion or recession in the economy. This implies that economic, political, financial and other factors that affect the economy also affects the income generated by the PFA. We thus assume that $Z(t)$ is given as the product of income growth rate $\xi(t)$ and the investment process $K(t)$. Hence,

$$dZ(t) = \xi(t)K(t)dt. \tag{23}$$

Equation (23) gives the income which accrues to the PFA for investing the amount $Z(t)$ in both financial and housing market at time t.

The income growth rate $\xi(t)$ is assumed to be risky and satisfies to the dynamics

$$d\xi(t) = g(\xi(t))dt + \sigma_Z dW(t), \xi(0) = \xi_0, \tag{24}$$

where $g(\xi(t)) : \mathbb{R} \times [0, T] \to \mathbb{R}$ is the expected growth of $\xi(t)$,
$\sigma_Z(t) = (\sigma_{Z_1}(t), \sigma_{Z_2}(t))$ is the volatility tensor of income growth rate such that $\sigma_{Z_1}(t) \in [\mathbb{R}^m \times [0, T]]$ is the volatiliy arising from uncertainity from inflation sources of risks and $\sigma_{Z_3}(t) \in [\mathbb{R}^q \times [0, T]]$ is the volatility arising from uncertatinity from housing prices risks at time t.

The net change in debt is given by

$$\begin{aligned} dL(t) &= r_L(t)L(t)dt + d\hat{Y}(t) + dC(t) - dZ(t), \\ &= r_L(t)L(t)dt + \hat{Y}(t)\big(\beta(t)dt + \sigma_A(t)dW(t) + \sigma_Y(t)dW_Y(t)\big) \\ &+ c(t)X(t)dt - \xi(t)K(t)dt. \end{aligned} \tag{25}$$

2.4 The nominal wealth process

In this subsection, we determine the nominal wealth process of the PFA at time t. But first, we give the following definition.

Definition 3. *The nominal wealth process X of the PFA at time t is defined as*

$$X(t) = K(t) - L(t). \tag{26}$$

.

Proposition 2. *The dynamics of the nominal wealth process of the PFA is*

$$\begin{aligned} dX = &\big[(r(t) + \xi(t) - c(t)) + \pi_L(t)(r(t) + \xi(t) - r_L(t)) - \hat{y}(t)\beta(t) - \pi_B(t)\sigma_B(t)\theta_B(t) \\ &- \pi_H(t)(r(t)\mathbf{I} - (1 + \pi_L(t) - k_F(t))(\mu_P(t) - \zeta(t)\mathbf{I})\big]dt - \hat{y}(t)\sigma_Y(t)dW_Y(t) \\ &+ \big(\pi_B(t)\sigma_B(t) + \pi_H(t)\sigma_P(t)(1 + \pi_L(t) - k_F(t)) - \hat{y}(t)\sigma_A(t)\big)dW(t), \\ &X(0) = x_0 \in \mathbb{R}_+. \end{aligned} \tag{27}$$

where
$\pi_H(t) = \frac{\Delta_H(t)}{X(t)}$ *is the proportion of wealth invested in hosing at time* t*,*
$\pi_L(t) = \frac{L(t)}{X(t)}$ *is the debt ratio at time* t*,*
$k_F(t) = \frac{K_F(t)}{X(t)}$ *is the fraction of wealth in the financial market at time* t*,*
$\pi_B(t) = \frac{\Delta_B(t)}{X(t)}$ *is the proportion of wealth invested in inflation-indexed bonds at time* t*.*

Proof. Taking the differential of both sides of (26), we have

$$dX(t) = dK(t) - dL(t). \tag{28}$$

Substituting (19) and (25) into (28), we have

$$\begin{aligned} dX &= \big[r(t)K(t) - \Delta_H(t)\big(r(t)\mathbf{I} - (K(t) - K_F(t))(\mu_P(t) - \zeta(t)\mathbf{I}\big) + \Delta_B(t)\sigma_B(t)\theta_B(t)\big]dt \\ &\quad + (\sigma_B(t)\Delta_B(t) + \Delta_H(t)\sigma_P(t)(K(t) - K_F(t)))dW(t) - r_L(t)L(t)dt - \hat{Y}(t)\big(\beta(t)dt \\ &\quad + \sigma_A(t)dW(t)\sigma_Y(t)dW_Y(t)\big) - c(t)X(t)dt + \xi(t)K(t)dt. \end{aligned} \tag{29}$$

using (26), we have the following

$$\begin{aligned} dX(t) &= \big[(r(t) + \xi(t) - c(t)) + \pi_L(t)(r(t) + \xi(t) - r_L(t)) - \hat{y}(t)\beta(t) - \pi_B(t)\sigma_B(t)\theta_B(t) \\ &\qquad - \pi_H(t)(r(t)\mathbf{I} - (1 + \pi_L(t) - k_F(t))(\mu_P(t) - \zeta(t)\mathbf{I})\big]dt - \hat{y}(t)\sigma_Y(t)dW_Y(t) \\ &\qquad + \big(\pi_B(t)\sigma_B(t) + \pi_H(t)\sigma_P(t)(1 + \pi_L(t) - k_F(t)) - \hat{y}(t)\sigma_A(t)\big)dW(t), \\ &\qquad X(0) = x_0 \in \mathbb{R}_+, X(0) = x_0 \in \mathbb{R}_+. \end{aligned} \tag{30}$$

□

Equation (30) represents the dynamics of the nominal wealth of the PFA at time t. However, the plan member is interested in the real value of his terminal wealth because of the impact of inflation. Let $\bar{X}$ be the real wealth process of the PFA at time t i.e nominal wealth adjusted for inflation.

Definition 4. *The real wealth* $\bar{X}(t)$ *of the PFA at time* t *is defined as the ratio of nominal wealth to inflation and is mathematically given by*

$$\bar{X}(t) = \frac{X(t)}{I(t)}. \tag{31}$$

Proposition 3. *The dynamics of real wealth of the PFA at time* t *is*

$$\begin{aligned} d\bar{X}(t) &= \left(\frac{X(t)}{I(t)}\right) = \bar{X}(t)\big[(r(t) + \xi(t) - c(t)) + \pi_L(t)(r(t) + \xi(t) - r_L(t)) - \hat{y}(t)\beta(t) \\ &\quad - \pi_B(t)\sigma_B(t)\theta_B(t) - \pi_H(t)(r(t)\mathbf{I} - (1 + \pi_L(t) - k_F(t))(\mu_P(t) - \zeta(t)\mathbf{I}) \\ &\quad - \big(\pi_B(t)\sigma_B(t) + \pi_H(t)\sigma_P(t)(1 + \pi_L(t) - k_F(t)) - \hat{y}(t)\sigma_A(t)\big)\sigma_I'(t) - \mu_I(t) \\ &\quad + \sigma_I(t)\sigma_I'(t) + \hat{y}(t)\rho_Y(t)\sigma_Y(t)\sigma_I'(t)\big]dt + \bar{X}(t)\big(\pi_B(t)\sigma_B(t) - \hat{y}(t)\sigma_A(t) - \sigma_I(t) \\ &\quad + \pi_H(t)\sigma_P(t)(1 + \pi_L(t) - k_F(t))\big)dW(t) - \bar{X}(t)\hat{y}(t)\sigma_Y(t)dW_Y(t), \bar{X}(0) = \bar{x}_0 \in \mathbb{R}_+ \end{aligned} \tag{32}$$

Proof. Using the Itô quotient rule on (31), where $I(t)$ and $X(t)$ satisfies (2) and (30) respectively, the result follows immediately.

For the purpose of simplicity, we shall omit the variable t except it becomes necessary to do so. It follows that

$$\begin{aligned} d\bar{X} = \left(\frac{X}{I}\right) &= \bar{X}\big[(r+\xi-c)+\pi_L(r+\xi-r_L)-\hat{y}\beta-\pi_B\sigma_B\theta_B- \\ &\pi_H(r\mathbf{I}-(1+\pi_L-k_F)(\mu_P-\zeta\mathbf{I})-\big(\pi_B\sigma_B+\pi_H\sigma_P(1+\pi_L-k_F)- \\ &\hat{y}\sigma_A\big)\sigma_I'-\mu_I+\sigma_I\sigma_I'+\hat{y}\rho_Y\sigma_Y\sigma_I'\big]dt+\bar{X}\big(\pi_B\sigma_B+\pi_H\sigma_P(1+\pi_L-k_F) \\ &-\hat{y}\sigma_A-\sigma_I\big)dW-\bar{X}\hat{y}\sigma_Y dW_Y, \bar{X}(0)=\bar{x}_0\in\mathbb{R}_+ \end{aligned} \tag{33}$$

where $\rho_Y = (\rho_{Y_1}, \rho_{Y_2})$ such that $\rho_{Y_1} \in [\mathbb{R}^m \times [0,T]$ and $\rho_{Y_2} \in [\mathbb{R}^q \times [0,T]$ is the correlation coefficient between inflation-indexed bonds, housing price risks and salary risk. □

3 The Optimal Investor's Choice

The PFA aim to select the optimal liability ratio, portfolio strategies and consumption rate that will optimize the expected discounted utility of consumption in an infinite time horizon. For an arbitrary admissible strategy $u = (\pi_B, \pi_H, \pi_L, c) : t \geq 0)$, the objective function $U(u, \bar{x}, \xi)$ follows:

$$U(u,\bar{x},\xi) = E_{\bar{x},\xi}\int_t^{\infty} e^{-\delta s}V(c(s)\bar{x}(s))ds, \tag{34}$$

where $V(c(s)\bar{x}(s))$ is utility function with respect to consumption and $0 \leq \delta < 1$ is the discount rate of consumption.

The strategy $u = \{\pi_B, \pi_H, \pi_L, c : t \geq 0\}$ that is progrssively measurable with respect to $\{W, W_Y : 0 \leq s \leq t\}$ is the admissible strategy. Let $\mathcal{A}$ be the collection of all admissible controls, then the collection of admissible controls $\mathcal{A}$ is defined as

$$\begin{aligned} &\mathcal{A} = \{u = (\pi_B,\pi_H,\pi_L,c : t\geq 0)\in\mathbb{R}^m\times\mathbb{R}^q\times\mathbb{R}\times\mathbb{R} : E\int_0^{\infty}\pi_B(t)\pi_B'(t)dt<\infty, \\ &E\int_0^{\infty}\pi_H(t)\pi_H'(t)dt<\infty, E\int_0^{\infty}c(t)^2(t)dt<\infty, E\int_0^{\infty}\pi_L^2(t)dt<\infty\}. \end{aligned} \tag{35}$$

We define our value function as

$$J(t,\bar{X}(t),\xi(t)) := \sup_{u\in\mathcal{A}}\Big[U(\bar{x},\xi)|\bar{X}(t)=\bar{x},\xi(t)=\xi\Big]. \tag{36}$$

The stochastic control problem is solved by adopting the dynamic programming approach to obtain the Hamilton-Jacobi-Bellman (HJB) equation. Solving the HJB equation gives the optimal values of our controls. The HJB equation characterizing the optimal solution to the PFA's problem is given as

$$\begin{aligned}
&J_t + \bar{x}\big[(r+\xi-c) + \pi_L(r+\xi-r_L) - \hat{y}\beta - \pi_B\sigma_B\theta_B - \\
&\pi_H(r\mathbf{I} - (1+\pi_L-k_F)(\mu_P-\zeta\mathbf{I}) - \big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \\
&\hat{y}\sigma_A\big)\sigma_I' - \mu_I + \sigma_I\sigma_I' + \hat{y}\rho_Y\sigma_Y\sigma_I'\big]J_{\bar{x}} + g(\xi)J_\xi + \frac{1}{2}\sigma_Z\sigma_Z' J_{\xi\xi} \\
&+\frac{1}{2}\bar{x}^2\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\big(\pi_B\sigma_B \\
&+\pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)' J_{\bar{x}\bar{x}} + \frac{1}{2}\bar{x}^2\hat{y}^2\sigma_Y^2 J_{\bar{x}\bar{x}} + \rho_1\bar{x}\hat{y}\sigma_Y\sigma_\xi J_{\bar{x}\xi} \\
&+\bar{x}\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\sigma_Z' J_{\bar{x}\xi} + e^{-\delta}V(c\bar{x}) = 0
\end{aligned} \tag{37}$$

with transvasility condition $\lim_{t\to\infty} \mathrm{E}[J(t,\bar{x},\xi)] = 0$, where ρ_1 is the correlation coefficient between income growth rate and salary process.

Using the standard time-homogeneity argument for problems with infinite-time horizon, we have

$$\begin{aligned}
e^{\delta t}J(t,\bar{X},\xi) &= \sup_{\{\pi_B(s),\pi_H(s),\pi_L(s),c(s)\}_{t\le s\le\infty}} \mathrm{E}_t \int_t^\infty e^{-\delta(s-t)}V(c(s)\bar{x}(s))ds \\
&= \sup_{\{\pi_B(t+u),\pi_H(t+u),\pi_L(t+u),c(t+u)\}_{0\le u\le\infty}} \mathrm{E}_t \int_0^\infty e^{-\delta(u)}V(c(t+u)\bar{x}(t+u))du \\
&= \sup_{\{\pi_B(u),\pi_H(u),\pi_L(u),c(u)\}_{0\le u\le\infty}} \mathrm{E}_0 \int_0^\infty e^{-\delta(u)}V(c(u)\bar{x}(u))du \\
&\equiv \mathrm{G}(t,\bar{X},\xi)
\end{aligned} \tag{38}$$

which is independent of time and the optimal control is Markov. Hence, $J(t,\bar{X},\xi) = e^{-\delta t}\mathrm{G}(t,\bar{X},\xi)$ and (38) reduces to the following time-homogeneous value function G:

$$\begin{aligned}
&\bar{x}\big[(r+\xi-c) + \pi_L(r+\xi-r_L) - \hat{y}\beta - \pi_B\sigma_B\theta_B - \\
&\pi_H(r\mathbf{I} - (1+\pi_L-k_F)(\mu_P-\zeta\mathbf{I}) - \big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \\
&\hat{y}\sigma_A\big)\sigma_I' - \mu_I + \sigma_I\sigma_I' + \hat{y}\rho_Y\sigma_Y\sigma_I'\big]\mathrm{G}_{\bar{x}} + g(\xi)\mathrm{G}_\xi + \frac{1}{2}\sigma_Z\sigma_Z' \mathrm{G}_{\xi\xi} \\
&+\frac{1}{2}\bar{x}^2\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\big(\pi_B\sigma_B \\
&+\pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)' \mathrm{G}_{\bar{x}\bar{x}} + \frac{1}{2}\bar{x}^2\hat{y}^2\sigma_Y^2 \mathrm{G}_{\bar{x}\bar{x}} + \rho_1\bar{x}\hat{y}\sigma_Y\sigma_\xi \mathrm{G}_{\bar{x}\xi} \\
&+\bar{x}\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\sigma_Z' \mathrm{G}_{\bar{x}\xi} + V(c\bar{x}) - \delta\mathrm{G}(t,\bar{X},\xi) = 0
\end{aligned} \tag{39}$$

with transvasility condition $\lim_{t\to\infty} \mathrm{E}[\mathrm{G}(t,\bar{x},\xi)] = 0$.

3.1 Power utility

Let $V(C) = \dfrac{C^{1-\gamma}}{1-\gamma}$ be the utility function of the investor with respect to consumption for $\gamma > 0$ such that $\gamma \in (0,1) \cup (1,\infty)$ and γ is a constant relative risk aversion parameter with respect to real wealth.

We assume that the solution to (39) is of the form

$$\mathrm{G}(t;\bar{x},\xi) = \frac{\bar{x}^{1-\gamma}}{1-\gamma} e^{h(\xi)}, \tag{40}$$

Differentiating (40) with respect to $\bar{x}, \bar{x}\bar{x}, \xi, \xi\xi$ and $\bar{x}\xi$, we have the following:

$$\begin{aligned} &\mathrm{G}_{\bar{x}} = \bar{x}^{-1}(1-\gamma)\mathrm{G}(t;\bar{x},\xi), \quad \mathrm{G}_{\bar{x}\bar{x}} = -\bar{x}^{-2}\gamma(1-\gamma)\mathrm{G}(t;\bar{x},\xi) \\ &\mathrm{G}_{\xi} = h_\xi \mathrm{G}(t;\bar{x},\xi), \quad \mathrm{G}_{\xi\xi} = (h_\xi^2 + h_{\xi\xi})\mathrm{G}(t;\bar{x},\xi) \\ &\mathrm{G}_{\bar{x}\xi} = h_\xi(1-\gamma)\bar{x}^{-1}\mathrm{G}(t;\bar{x},\xi). \end{aligned} \tag{41}$$

Substituting (41) into (39), we have

$$\begin{aligned} 0 = &\max_{\{\pi_B,\pi_H,\pi_L,c\}} \{\big[(r+\xi-c) + \pi_L(r+\xi-r_L) - \hat{y}\beta - \pi_B\sigma_B\theta_B \\ &- \pi_H(r\mathbf{I} - (1+\pi_L-k_F)(\mu_P - \zeta\mathbf{I}) - \big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \\ &\hat{y}\sigma_A\big)\sigma_I' - \mu_I + \sigma_I\sigma_I' + \hat{y}\rho_Y\sigma_Y\sigma_I'\big](1-\gamma)\mathrm{G}(t;\bar{x},\xi) + g(\xi)h_\xi\mathrm{G}(t;\bar{x},\xi) \\ &+ \frac{1}{2}\sigma_Z\sigma_Z'(h_{\xi\xi}+h_\xi^2)\mathrm{G}(t;\bar{x},\xi) - \frac{1}{2}\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big) \\ &\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)'\gamma(1-\gamma)\mathrm{G}(t;\bar{x},\xi) - \frac{1}{2}\hat{y}^2\sigma_Y^2\gamma(1-\gamma)\mathrm{G}(t;\bar{x},\xi) \\ &+ V(c\bar{x}) - \delta\mathrm{G}(t;\bar{x},\xi) + \big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\sigma_Z' h_\xi(1-\gamma)\mathrm{G}(t;\bar{x},\xi) \\ &+ \rho_1\bar{x}\hat{y}\sigma_Y\sigma_\xi h_\xi(1-\gamma)\mathrm{G}(t;\bar{x},\xi)\}. \end{aligned} \tag{42}$$

We now divide through (42) by $-(1-\gamma)\mathrm{G}^{\bar{x}} < 0$, so that max becomes min. Hence, we have

$$\begin{aligned} 0 = &\min_{\{\pi_B,\pi_H,\pi_L,c\}} \{-\big[(r+\xi-c) + \pi_L(r+\xi-r_L) - \hat{y}\beta - \pi_B\sigma_B\theta_B \\ &- \pi_H(r\mathbf{I} - (1+\pi_L-k_F)(\mu_P - \zeta\mathbf{I}) - \big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \\ &\hat{y}\sigma_A\big)\sigma_I' - \mu_I + \sigma_I\sigma_I' + \hat{y}\rho_Y\sigma_Y\sigma_I'\big] - g(\xi)\frac{h_\xi}{(1-\gamma)} - \frac{1}{2(1-\gamma)}\sigma_Z\sigma_Z'(h_{\xi\xi}+h_\xi^2) \\ &+ \frac{1}{2}\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)'\gamma \\ &+ \frac{1}{2}\hat{y}^2\sigma_Y^2\gamma - \frac{V(c\bar{x})}{(1-\gamma)\mathrm{G}(t;\bar{x},\xi)} + \frac{\delta}{1-\gamma} \\ &- \big(\pi_B\sigma_B + \pi_H\sigma_P(1+\pi_L-k_F) - \hat{y}\sigma_A - \sigma_I\big)\sigma_Z' h_\xi - \rho_1\bar{x}\hat{y}\sigma_Y\sigma_\xi h_\xi\}. \end{aligned} \tag{43}$$

Considering (53), we obtain the optimal debt ratio, optimal portfolio strategies and optimal consumption rate.

3.2 Optimal debt ratio of the PFA

Definition 5. *The optimal debt ratio of the investor* ϕ^* *is defined as*

$$\pi_L^* = arg\min_{\pi_L} f(\pi_L),$$

where the function

$$\begin{aligned} f(\pi_L) = & -\pi_L(r+\xi-r_L) + \frac{1}{2}\pi_H\sigma_P(\pi_H\sigma_P)'(1+\pi_L-k_F)^2\gamma + \pi_L\big((\mu_P-\zeta\mathbf{I})\pi_H \\ & +\sigma_P\pi_H\sigma_I'(1-\gamma) - \sigma_P\pi_H\sigma_Z' h_\xi - \sigma_P\pi_H\hat{y}\sigma_A'\gamma + \sigma_P\pi_H(\pi_B\sigma_B)'\gamma\big). \end{aligned} \tag{44}$$

is convex.

Proposition 4. *The optimal debt ratio of the PFA is given by*

$$\begin{aligned} \pi_L^*(t) = & -(1-k_F) + \frac{(r+\xi-r_L)}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} - \frac{(\mu_P-\zeta\mathbf{I})}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} - \frac{\sigma_P\pi_H^*\sigma_I'(1-\gamma)}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} \\ & + \frac{\sigma_P\pi_H^*\sigma_Z' h_\xi}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} + \frac{\sigma_P\pi_H^*\hat{y}\sigma_A'}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'} - \frac{\sigma_P\pi_H^*(\pi_B^*\sigma_B)'}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'}. \end{aligned} \tag{45}$$

.

Proof. By first order condition, we have from (44) that

$$\begin{aligned} \frac{\partial f(\pi_L)}{\partial \pi_L} = & -(r+\xi-r_L) + \pi_H\sigma_P(\pi_H\sigma_P)'(1+\pi_L-k_F)\gamma + (\mu_P-\zeta\mathbf{I})\pi_H \\ & +\sigma_P\pi_H\sigma_I'(1-\gamma) - \sigma_P\pi_H\sigma_Z' h_\xi - \sigma_P\pi_H\hat{y}\sigma_A'\gamma + \sigma_P\pi_H(\pi_B\sigma_B)'\gamma = 0. \end{aligned} \tag{46}$$

This implies that

$$\begin{aligned} \pi_L^*(t) = & -\underbrace{\frac{\sigma_P\pi_H^*\sigma_I'(1-\gamma)}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} + \frac{\sigma_P\pi_H^*\sigma_Z' h_\xi}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} + \frac{\sigma_P\pi_H^*\hat{y}\sigma_A'}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'} - \frac{\sigma_P\pi_H^*(\pi_B^*\sigma_B)'}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'}}_{\beta_1} \\ & \underbrace{-(1-k_F)}_{\beta_2} + \underbrace{\frac{(r+\xi-r_L)}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma} - \frac{(\mu_P-\zeta\mathbf{I})}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'\gamma}}_{\beta_3} \end{aligned} \tag{47}$$

□

We observe that the optimal debt ratio is made

Corollary 1. *Suppoe the PFA becomes increasingly risk averse i.e* $\gamma \to \infty$*, then*

$$\pi_L^*(t) = -(1-k_F) + \chi\Big[\sigma_P\pi_H^* + \hat{y}\sigma_A' - (\pi_B^*\sigma_B)'\Big]. \tag{48}$$

where $\chi = \dfrac{\sigma_I'}{(\pi_H^*\sigma_P)(\pi_H^*\sigma_P)'}$.

3.3 Optimal portfolio policy for housing

Definition 6. *The optimal portfolio strategy of the investor in housing π_H^* is defined as*

$$\pi_H^* = arg\min_{\pi_H} f(\pi_H),$$

where the function

$$\begin{aligned} f(\pi_H) &= \pi_H(r\mathbf{I} - (1+\pi_L - k_F)(\mu_P - \zeta\mathbf{I})) + \pi_H\sigma_P(1+\pi_L-k_F)\sigma_I' \\ &-\pi_H\sigma_P(1+\pi_L-k_F)\sigma_Z' h_\xi + \frac{1}{2}\pi_H\sigma_P(\pi_H\sigma_P)'(1+\pi_L-k_F)^2\gamma \\ &+\pi_B\sigma_B(\pi_H\sigma_P)'(1+\pi_L-k_F)\gamma - \pi_H\sigma_P(1+\pi_L-k_F)\hat{y}\sigma_A'\gamma. \end{aligned} \tag{49}$$

is convex.

Proposition 5. *The optimal portfolio strategy in housing is given by*

$$\begin{aligned} \pi_H^* &= \frac{\sigma_P\hat{y}\sigma_A'}{\Psi} - \frac{\sigma_P(\pi_B\sigma_B)}{\Psi} - \frac{\sigma_P\sigma_I'(1-\gamma)}{\gamma\Psi} + \frac{\sigma_P\sigma_Z' h_\xi}{\Psi\gamma} \\ &+ \frac{1}{\gamma\Psi}\left[(\mu_P - \zeta\mathbf{I}) - \frac{r\mathbf{I}}{(1+\pi_L-k_F)}\right] \end{aligned} \tag{50}$$

where $\Psi = (\sigma_P\sigma_P')(1+\pi_L-k_F)$

Proof. By first order condition, we have that

$$\begin{aligned} \frac{\partial f_H(\pi_H)}{\partial \pi_H} &= (r\mathbf{I} - (1+\pi_L-k_F)(\mu_P - \zeta\mathbf{I})) + \sigma_P(1+\pi_L-k_F)\sigma_I' \\ &- \sigma_P(1+\pi_L-k_F)\sigma_Z' h_\xi + \sigma_P(\pi_H\sigma_P)'(1+\pi_L-k_F)\gamma \\ &+ \pi_B\sigma_B\sigma_P'(1+\pi_L-k_F)\gamma - \sigma_P(1+\pi_L-k_F)\hat{y}\sigma_A'\gamma = 0 \end{aligned} \tag{51}$$

Making π_H^* the subject of the formular, we have

$$\begin{aligned} \pi_H^* &= \underbrace{\frac{\sigma_P\hat{y}\sigma_A'}{\Psi} - \frac{\sigma_P(\pi_B\sigma_B)}{\Psi} - \frac{\sigma_P\sigma_I'(1-\gamma)}{\gamma\Psi} + \frac{\sigma_P\sigma_Z' h_\xi}{\Psi\gamma}}_{\psi_1} \\ &+ \underbrace{\frac{1}{\gamma\Psi}\left[(\mu_P - \zeta\mathbf{I}) - \frac{r\mathbf{I}}{(1+\pi_L-k_F)}\right]}_{\psi_2}. \end{aligned} \tag{52}$$

□

(52) represents the optimal portfolio weight of the PFA in housing. The housing portfolio has two components: The speculative component ψ_2 adjusted for depreciation in the housing asset through the inverse of $1 + \pi_L - k_F$ and the relative risk averse index $\frac{1}{\gamma}$. The hedging portfolio component ψ_2 reflects the investor's desire to hedge against fluctuations in salary contribution, investment portfolio, inflation and income growth rate risks.

3.4 Optimal portfolio policy for inflation-linked bonds

Definition 7. *The optimal portfolio strategy of the investor for inflation-linked bonds* π_B^* *is defined as*

$$\pi_B^* = arg\min_{\pi_B} f(\pi_B),$$

where the function

$$f(\pi_B) = \pi_B\sigma_B\theta_B + \pi_B\sigma_B\sigma_I'(1-\gamma) + \frac{1}{2}\pi_B\sigma_B(\pi_B\sigma_B)'\gamma + \pi_B\sigma_B(1+\pi_L-k_F)(\pi_H\sigma_P)'\gamma \\ -\hat{y}\pi_B\sigma_B\sigma_A'\gamma - \pi_B\sigma_B\sigma_Z'h_\xi. \tag{53}$$

is convex.

Proposition 6. *The optimal portfolio strategy in indexed bond is given by*

$$\pi_B^* = -\frac{\Sigma_B^{-1}}{\gamma}\sigma_B(\sigma_I'+\theta_B) - \Sigma^{-1}\pi_S^*\sigma_S\sigma_B' - \Sigma_B^{-1}\pi_H^*\sigma_P\sigma_B' + \Sigma_B^{-1}\hat{y}\sigma_A\sigma_B' \\ +\frac{\Sigma_B^{-1}}{\gamma}\sigma_B h_\xi(\sigma_\xi\rho_\xi - \sigma_Z') \tag{54}$$

Proof. By first order condition, we have that

$$\frac{\partial f_B(\pi_B)}{\partial \pi_B} = \sigma_B\theta_B + \sigma_B\sigma_I'(1-\gamma) + \sigma_B(\pi_B\sigma_B)'\gamma + \sigma_B(1+\pi_L-k_F)(\pi_H\sigma_P)'\gamma \\ -\hat{y}\sigma_B\sigma_A'\gamma - \sigma_B\sigma_Z'h_\xi = 0 \tag{55}$$

Making π_B^* the subject of the formular, we have

$$\pi_B^* = \underbrace{-\frac{1}{\gamma}\Sigma_B^{-1}\sigma_B\theta_B}_{\zeta_1} - \underbrace{\frac{(1-\gamma)}{\gamma}\Sigma_B^{-1}\sigma_B\sigma_I'}_{\zeta_2} + \underbrace{\frac{1}{\gamma}\Sigma_B^{-1}\sigma_B\sigma_Z'h_\xi}_{\zeta_3} + \underbrace{\Sigma_B^{-1}\hat{y}\sigma_B\sigma_A'}_{\zeta_4} \\ -\underbrace{\sigma_B(1+\pi_L-k_F)(\pi_H\sigma_P)'\Sigma_B^{-1}}_{\zeta_5}. \tag{56}$$

□

Here (56) is the optimal portfolio strategy of the investor in inflation-linked bonds. Clearly, the portfolio strategy in inflation-linked bond has hedge components for salary, housing, income growth rate and inflation risks. It is therefore optimal for the PFA to invest in an inflation-linked bond portfolio with the following composition:

1. an inflation risk hedging portfolio ζ_1 proportional to the diffusion term of inflation index, the market price of risk and the relative risk aversion coefficient $\frac{1}{\gamma}$,

2. a stock market risk hedging strategy ζ_2 proportional to the volatility of the stock price and the portfolio policy in stock,

3. a housing risk hedging strategy ζ_3 proportional to the volatility of the housing price and the portfolio policy in housing,

4. a hedge portfolio ζ_4 against salary risk, which is proportional to the volatility of the salary process and the contribution process,

5. an income risk hedging strategy ζ_5 proportional to the volatility of income growth rate, through the correlation coefficient between housing assets and income growth rate, the cross derivative of h with respect to ξ and the relative risk averse index $\frac{1}{\gamma}$.

3.5 Optimal consumption rate

In this subsection, we determine the optimal consumption rate of the investor. For the optimal consumption rate, we have that

$$\frac{\partial V(c)}{\partial c} = (1-\gamma)\mathrm{G}^{\bar{x}}. \tag{57}$$

This implies that

$$c^*(t) = \left[\bar{x}(t)^{1-\gamma} e^{h(t,\xi)}\right]^{-\frac{1}{\alpha}}. \tag{58}$$

It follows that terminal consumption (i.e for t=T) is

$$c^*(T) = \left[\bar{x}(t)^{1-\gamma}\right]^{-\frac{1}{\alpha}}. \tag{59}$$

Since $h(t,\xi) = 0$ as shown earlier. At $t = 0$, we have that

$$c^*(0) = \left[\bar{x}_0^{1-\gamma} e^{h(0,\xi)}\right]^{-\frac{1}{\alpha}}. \tag{60}$$

4 Empirical Results

In this section, we provide numerical results for our models. For the purpose of relating the results generated to pratical issues in finance and the economy, we collected average housing price data for four states in Nigeria and bond price data for four bonds between Feburary 2018 to June 2023. The data for average housing price for four states include: HA1, HA2, HA3 and HA4 and also, four government bonds, referred here to as: BOND1, BOND2, BOND3 and BOND4 were collected from DMO website. We analyse the collected data to obtain the necessary information statistic (mean, standard deviation and covariance matrix) required for implementing our model. Using the salary chart of Nigerian University workers of 2009 since that is what is still being operated in Nigerian universities. We calculated the mean and volatility of the PPM's salary. The salary chart was obtained from the University of Benin, Bursary Department. We analysed the data using Statistical Package for the Social Sciences (SPSS). The inflation yearly average data collected from the International Monetary Fund between 1980 to 2023 were analyzed. The mean and volatility for the inflation rate data were calculated as 16.6957% and 18.7755% respectively. Table 1 shows the mean and Volatility for bonds, stocks and housing assets respectively.

Table 1: Indexed bonds and housing assets statistics

Stocks	Mean	volatility	Housing	Mean	volatility
BOND1	0.7689	0.0712	HA1	03.4558	0.0829
BOND2	0.9073	0.0757	HA2	5.0639	0.0157
BOND3	1.1169	0.1631	HA3	2.7704	0.0663
BOND4	0.7697	0.0789	HA4	4.7701	0.0433

The covariance matrix for the bonds is

$$\sigma_B = \begin{pmatrix} 0.0770 & 0.0120 & -0.0720 & -0.0070 \\ 0.0120 & 0.1840 & -0.0940 & -0.0830 \\ -0.0720 & -0.0940 & 0.2190 & -0.1000 \\ -0.0070 & -0.0830 & -0.1000 & 0.2300 \end{pmatrix}$$

The covariance matrix for the housing assets is

$$\sigma_H = \begin{pmatrix} 0.0714 & 0.0026 & -0.0014 & -0.0288 \\ 0.0026 & 0.0035 & -0.0006 & -0.0088 \\ -0.0014 & -0.0006 & 0.0058 & 0.0221 \\ -0.0288 & -0.0088 & 0.0221 & 0.1425 \end{pmatrix}$$

Figure 1 shows the debt ratio of the PFA against interest rate on debt over time. We observe that the higher the interest rate paid on debt, the lower the debt ratio of the PFA. This clearly indicates that to reduce debt burden, the PFA needs to increase the amount paid for servicing loans collected. Figure 2 shows the debt ratio for a given wealth and coefficient of risk aversion γ for $\varphi = 0.05$. We observe that as the wealth of the PFA increases, the debt ratio decreases. This indicates from the PFA's persperctive, that the greater the wealth, the lower the burden of debt and the greater their ability to meet her obligations to the PPM. Also, as γ increases, the debt ratio increases as well.

We see from Figure 3 that with an increase in the CRRA coefficient w.r.t. debt for $\varphi = 0.5$, there is a reduction in the optimal debt ratio and the debt ratio also becomes less sensitive to increase in real wealth. This implies that a PFA with a high CRRA coefficient w.r.t. debt will accumulate less debt.

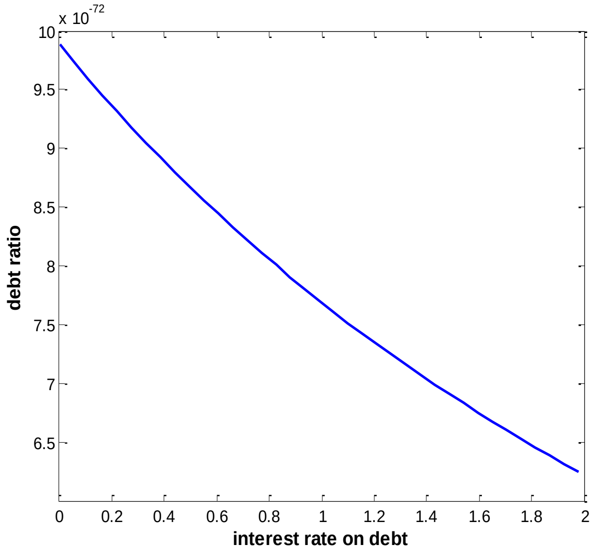

Figure 1: Debt ratio plotted against interest rate on debt for $r = 0.045$, $\gamma = 0.5$, $\bar{x} = 100$ and $\varphi = 0.05$.

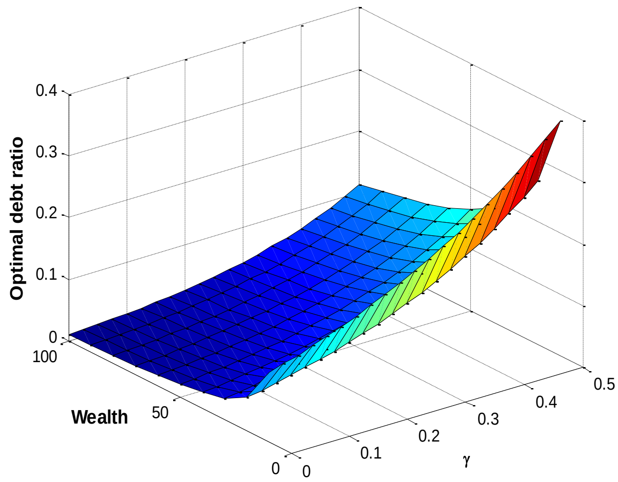

Figure 2: Debt ratio plotted against γ for $r_L = 0.065$, $\varphi = 0.05$ and $r = 0.045$.

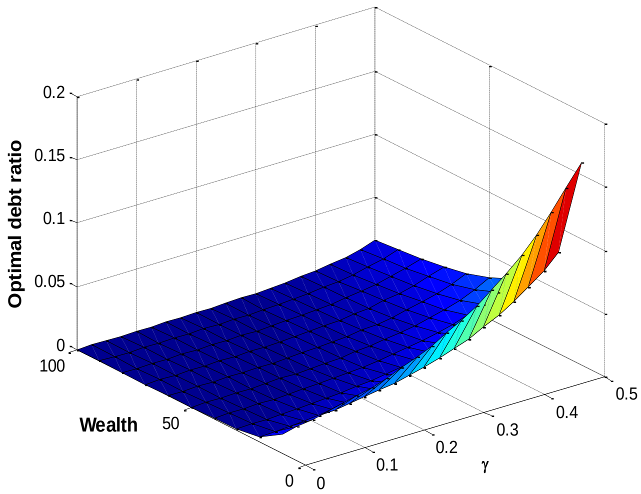

Figure 3: Debt ratio plotted against φ for $r_L = 0.065$, $\varphi = 0.5$ and $r = 0.045$.

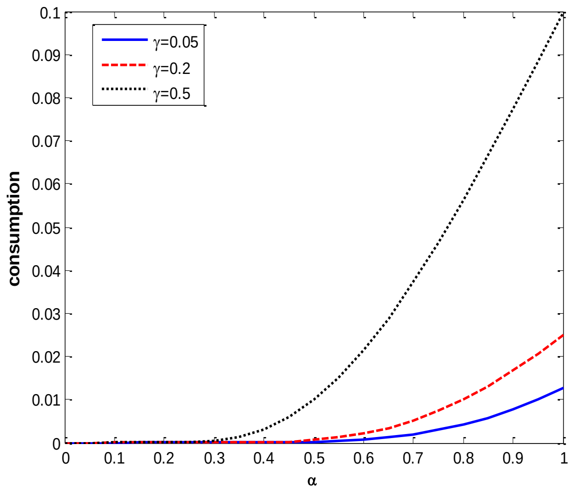

Figure 4: Effects of the risk aversion coefficients with respect to consumption and real wealth on optimal consumption.

Table 2: The optimal portfolio values for varying $\hat{y}$

$\hat{y}$	π_0^*	Bonds	π_B^*	Housing	π_H^*
0.2	-14.5564	BOND1	12.1089	HA1	465.9811
		BOND2	3.8326	HA2	137.3668
		BOND3	0.4643	HA3	206.4103
		BOND4	21.4010	HA4	-810.0049
0.4	-30.3441	BOND1	14.6552	HA1	464.0026
		BOND2	4.5833	HA2	136.7854
		BOND3	6.5952	HA3	205.5327
		BOND4	25.8307	HA4	-806.6144
0.6	-41.1618	BOND1	17.2016	HA1	462.0240
		BOND2	5.3340	HA2	136.2039
		BOND3	7.7261	HA3	204.6551
		BOND4	30.2603	HA4	-803.2239
0.8	-51.9193	BOND1	19.7479	HA1	460.0455
		BOND2	6.0847	HA2	135.6224
		BOND3	8.8570	HA3	203.7774
		BOND4	34.6900	HA4	-799.8334

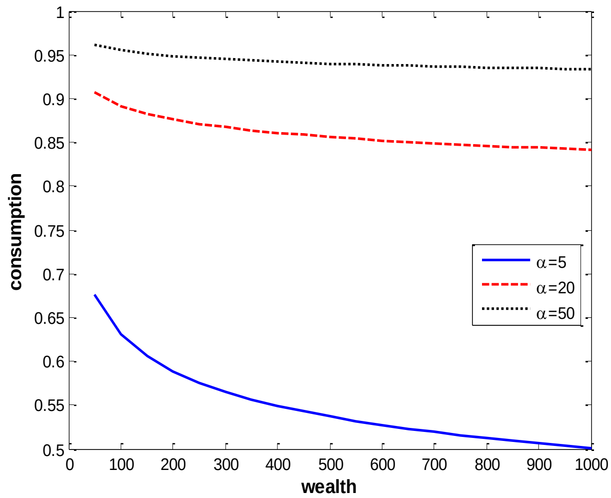

Figure 5: Effects of real wealth and risk aversion coefficient with respect to consumption on optimal consumption.

Figure 4 plots the effects of risk aversion coefficients w.r.t. consumption and real wealth on optimal consumption. We observe that optimal consumption increases w.r.t. α and γ. Figure 5 demonstrates the effects of real wealth and CRRA coefficient w.r.t. consumption on the optimal consumption of the PFA. We find from Figure 5 that optimal consumption decreases with an increase in real wealth and increases with an increase in the CRRA coefficient w.r.t. consumption.

Table 2 shows the PFA's optimal portfolio returns for varying values of the contribution of the plan member for other parameters remain fixed. It is observed that as the contribution of the PPM increases, portfolio values in stocks and indexed bonds increases as well. However, the risk-free portfolio and housing asset decreases except for HA4 that increases with an increase in $\hat{y}$. This shows that the PFA should invest more in indexed bonds, stocks and HA4 as the contributions of plan members increase.

5 Concluding Remarks

We now provide the concluding remarks for this paper. We have considered the optimal investment strategies, debt ratio and consumption plan for a DC pension scheme. The model was studied under five background risks that included inflation, stock, housing, income rate and salary risks. The asset class for this study was divided into two: housing and financial assets. The optimal debt ratio, optimal portfolio strategies and optimal consumption plan of the PFA under the CRRA utility function were obtained. We found that the optimal debt ratio depends on interest rate paid on debt, the nominal interest rate, CRRA coefficient with respect to real wealth and debt, the optimal wealth and income growth rate. The optimal portfolio strategies were found to made up of five components and their explicit form were also obtained. From the numerical analysis the following were found: (i) that the optimal debt ratio is positively related to the risk aversion coefficient with respect to real wealth but inversely to the risk aversion coefficient with respect to debt, (ii) that optimal consumption is positively related to the CRRA coefficients w.r.t. consumption and real wealth, (iii) with an increase in contribution by the PPM, investment in stocks and indexed bonds increase but the riskless asset and housing experience a decrease expect for HA4.

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