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Bi-univalent Function Subfamilies Associated with the (p, q)-derivative Operator Subordinate to Lucas-Balancing Polynomials

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Abstract

In the open disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\} = \mathfrak{D}$, we present a family of bi-univalent functions $g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j$ associated with the (p, q)-derivative operator and Lucas-Balancing polynomials. For members of this family, we obtain the upper bounds for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$. The new implications of the main results are also discussed, along with relevant connections to earlier research.

1 Preliminaries

A generalization of the ordinary calculus without the use of limit concepts is the q-calculus. Jackson presented the use and application of the q-calculus in [29]. The extension of the q-calculus to the

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(p,q)-calculus was taken into account by the researchers. At about the same time, in 1991, the (p,q)-number and the (p,q)-calculus were first analyzed by Arik [6], Brod [11], Chakrabarti [16], Wach [54], and others. In [6], the (p,q)-number was presented in order to study Fibonacci oscillators. A (p,q)-Harmonic oscillator can be constructed thanks to the (p,q)-number investigation in [11]. In [16], the (p,q)-number was studied to unify various q-oscillator algebra types. In [54], the (p,q)-numbers are examined in order to determine the (p,q)-Stirling numbers. Building on the aforementioned publications, since 1991, a large number of researchers have investigated the (p,q)-calculus in a range of scientific domains. The results presented in [30] provided an embedding syntax for q-series into a (p,q)-series. Moreover, they discovered some outcomes that matched (p,q)-extensions of the established q-identities. A corresponding extension of the q-identities yields the (p,q)-series (see, for instance, [5]). We give some basic explanations of the concepts in (p,q)-calculus. $[j]_{p,q} = p^{j-1} + p^{j-2}q + \cdots + p^2q^{j-3} + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p-q}$, $0 < q < p \leq 1$, is the formula for the (p,q)-bracket number, which is an extension of q-number $\frac{1-q^j}{1-q} = [j]_q$ $(q \neq 1)$(refer to [28]). We remark that $[j]_{p,q}$ is symmetric and that $[j]_{p,q}$=$[j]_q$ if $p = 1$.

Definition 1.1. [52] The (p,q)-derivative of φ is defined by

$$D_{p,q}\varphi(\zeta) = \frac{\varphi(p\zeta) - \varphi(q\zeta)}{(p-q)\zeta} \ (\zeta \neq 0), \text{ and } D_{p,q}\varphi(0) = \varphi'(0), \text{ provided } \varphi'(0) \text{ exists,}$$

where the function φ is defined on the complex plane $\mathbb{C}$ and $0 < q < p \leq 1$.

We know that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$ and $D_{p,q}\log(\zeta) = \frac{\log(p/q)}{(p-q)\zeta}$. If $p = 1, q \to 1^-$, then $[j]_{p,q} \to j$, and $D_{p,q}\varphi(\zeta) \to \varphi'(\zeta)$, as well. In particular, $D_{p,q}(a\varphi_1(\zeta) + b\varphi_2(\zeta)) = aD_{p,q}\varphi_1(\zeta) + bD_{p,q}\varphi_2(\zeta)$, where a and b are constants. The quotient and product rules are satisfied by the (p,q)-derivative (see [39]). The (p,q)-analogues of some trigonometric functions are defined using the exponential functions [12, 19, 52].

Consider the normalized analytic function θ in $\{\zeta \in \mathbb{C} : |\zeta| < 1\} = \mathfrak{D}$, which is provided by

$$\theta(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j\zeta^j, \tag{1.1}$$

and let $\mathcal{A}$ be the class of all such functions. Let $\mathcal{S} = \{\theta \in \mathcal{A} : \theta \text{ is univalent in } \mathfrak{D}\}$. If $\theta \in \mathcal{A}$ is the kind (1.1), then

$$D_{p,q}\theta(\zeta) = 1 + \sum_{j=2}^{\infty}[j]_{p,q}d_j\zeta^{j-1}, \quad (\zeta \in \mathfrak{D}), \tag{1.2}$$

According to the widely recognized Koebe result (see [20]), every function $\theta \in \mathcal{S}$ has an inverse, which is given by

$$\theta^{-1}(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots = \Theta(\omega), \tag{1.3}$$

satisfying $\zeta = \theta^{-1}(\theta(\zeta))$, $\omega = \theta(\theta^{-1}(\omega))$, $|\omega| < r_0(\theta), r_0(\theta) \geq 1/4$, and ζ, $\omega \in \mathfrak{D}$.

Levin first introduced the concept of bi-univalent functions in his investigation [31]. These are functions $\theta \in \mathcal{A}$, where θ and θ^{-1} are both univalent in $\mathfrak{D}$, and Σ represents the set of all bi-univalent functions. For example, $\frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right)$, $\frac{\zeta}{1-\zeta}$ and $-\log(1-\zeta)$ are elements in the Σ family. That being said, even though $\mathcal{S}$ contains $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function, they do not belong in Σ. Refer to [9, 10, 51] and the citation given in these papers for a succinct study and to learn about the traits of the set Σ. The results of Srivastava et al. [42] triggered the recent surge in research on the bi-univalent function family. Several intriguing special families of Σ have been studied by numerous researchers since this article revived the topic (see [13, 14, 22]).

Several subfamilies of the family Σ were studied using the (p, q)-calculus. The (p, q)-starlike and (p, q)-convex function families are explored in [43] using the subordination principle. Many studies have also been presented and investigated certain Σ subfamilies defined using the (p, q)-differential operators (see [2, 3, 17, 53]).

The focus at the moment is on functions that are subordinate to known special polynomials and belong to a specific σ family. See [1, 23, 44, 47, 49, 50] for further details on these. The Lucas-Balancing polynomials are one type of these polynomials that has caught the interest of researchers.

C_j represents the Balancing numbers (BN), which satisfy the recurrence relation (RR) (see [7])

$$C_{j+1} = 6C_j - C_{j-1}, \quad (C_0 = 1, C_1 = 1, j \geq 1).$$

A Lucas-Balancing numbers (LBN) sequence is $B_j = \sqrt{8C_j^2 + 1}, j \geq 1$. It satisfies the RR $B_{j+1} = 6B_j - B_{j-1}, j \geq 1$, with $B_0 = 1$ and $B_1 = 3$. These numbers have been examined in [18, 25, 26, 35, 37]. Naturally occurring extensions of BN and LBN are known as Balancing polynomials (BP) and Lucas-Balancing polynomials (LBP), respectively. The recursive definition of BP, represented by $C_j(\varkappa)$, is

$$C_j(\varkappa) = 6\varkappa C_{j-1}(\varkappa) - C_{j-2}(\varkappa), \quad (C_0(\varkappa) = 0, C_1(\varkappa) = 1, j \geq 2),$$

where $\varkappa \in \mathbb{C}$. It is evident that $C_2(\varkappa) = 6\varkappa$, $C_3(\varkappa) = 36\varkappa^2 - 1$, $C_4(\varkappa) = 216\varkappa^3 - 12\varkappa$, and so forth. The recursive definition of LBP, denoted by $B_j(\varkappa), \varkappa \in \mathbb{C}$, is given by

$$\mathsf{B}_j(\varkappa) = 6\varkappa \mathsf{B}_{j-1}(\varkappa) - \mathsf{B}_{j-2}(\varkappa), \quad (j \geq 2, \mathsf{B}_0(\varkappa) = 1,\ \mathsf{B}_1(\varkappa) = 3\varkappa). \tag{1.4}$$

$\mathsf{B}_2(\varkappa) = 18\varkappa^2 - 1, \mathsf{B}_3(\varkappa) = 108\varkappa^3 - 9\varkappa, \ldots$ are evident from (1.4). To learn more about this field, researchers can visit [8, 32, 36, 38]. According to [24], the generating function (GF) of the LBP is represented by

$$\mathsf{B}(\varkappa, \zeta) := \sum_{j=0}^{\infty} \mathsf{B}_j(\varkappa)\zeta^j = \frac{1 - 3\varkappa\zeta}{1 - 6\varkappa\zeta + \zeta^2}, \tag{1.5}$$

where $\varkappa \in [-1, 1]$, and $\zeta \in \mathfrak{D}$.

Definition 1.2. For $\theta \in \mathcal{A}$, the (p,q)-analogue of the Swamy operator [45] is defined as follows:

$$W_{p,q}^{\nu,\mu,0}\theta(\zeta) = \theta(\zeta),$$

$$W_{p,q}^{\nu,\mu,1}\theta(\zeta) = \frac{\nu\theta(\zeta) + \mu z D_{p,q}\theta(\zeta)}{\nu+\mu}, \cdots,$$

$$W_{p,q}^{\nu,\mu,k}\theta(\zeta) = W_{p,q}^{\nu,\mu}(W_{p,q}^{\nu,\mu,k-1}\theta(\zeta)),$$

where $0 < q < p \leq 1, k \in \mathbb{N}, \mu \geq 0,\ \nu \in \mathbb{R}$ with $\nu + \mu > 0,$ and $\zeta \in \mathfrak{D}$.

Remark 1.1. i). For $\theta(\zeta)$ given by (1.1), we have

$$W_{p,q}^{\nu,\mu,k}\theta(\zeta) = \zeta + \sum_{j=2}^{\infty}\left(\frac{\mu[j]_{p,q}+\nu}{\mu+\nu}\right)^k d_j\zeta^j, \tag{1.6}$$

ii). The operator $W_{p,q}^{\nu,\mu,k}$ reduces to the (p,q)-analogue of the operator described by Selvaraj et al. in [41] if we assume that $\nu = 0$ and $\mu = 1$.

iii). If $\nu = 1-\mu, \mu \geq 0$, then $A_{p,q}^{\mu,k} : \mathcal{A} \to \mathcal{A}$ is (p,q)-analogue of Al-Oboudi differential operator, where $A_{p,q}^{\mu,k} = W_{p,q}^{1-\mu,\mu,k}$ and for $\theta(\zeta)$ given by (1.1), we have

$$A_{p,q}^{\mu,k}\theta(\zeta) = \zeta + \sum_{j=2}^{\infty}\left(\mu([j]_{p,q}-1)+1\right)^k d_j\zeta^j. \tag{1.7}$$

iv). If $\nu = l+1-\mu, l > -1, \mu \geq 0$, then $C_{p,q}^{l,\mu,k} : \mathcal{A} \to \mathcal{A}$ is Catas operator's (p,q)-analogue, where $C_{p,q}^{l,\mu,k} = W_{p,q}^{l+1-\mu,\mu,k}$ and for $\theta(\zeta)$ given by (1.1), we have

$$C_{p,q}^{l,\mu,k}\theta(\zeta) = \zeta + \sum_{j=2}^{\infty}\left(\frac{\mu([j]_{p,q}-1)+l+1}{l+1}\right)^k d_j\zeta^j. \tag{1.8}$$

v). Three operators defined in [45], [4], and [15] are obtained by taking $q \to 1^-$ and $p = 1$ in (1.6), (1.7), and (1.8), respectively. Swamy operator is extended to k-valent functions in [46].

For functions ϑ_1, $\vartheta_2 \in \mathcal{A}$, ϑ_1 is subordinate to ϑ_2, if there is a Schwarz function $\kappa(\zeta)$ in $\mathfrak{D}$ with $\kappa(0) = 0$ and $|\kappa(\zeta)| < 1$, with $\vartheta_1(\zeta) = \vartheta_2(\kappa(\zeta)), \zeta \in \mathfrak{D}$. This is shown as $\vartheta_1 \prec \vartheta_2$ or $\vartheta_1(\zeta) \prec \vartheta_2(\zeta)$ $(\zeta \in \mathfrak{D})$. Especially, if $\vartheta_2 \in \mathcal{S}$, then $\vartheta_1(\zeta) \prec \vartheta_2(\zeta) \Leftrightarrow \vartheta_1(0) = \vartheta_2(0)$ and $\vartheta_1(\mathfrak{D}) \subset \vartheta_2(\mathfrak{D})$.

We propose a subfamily of Σ using the (p,q)-analogue of the Swamy derivative operator, subordinate to LBP $B_j(\varkappa)$ as in (1.4) with the GF as in (1.5). This subfamily is inspired by the recent developments on functions $\in \Sigma$ related to LBP [27, 48].

Let $\mathbb{N} := \{1,2,3,\cdots\}$ and $\mathbb{R} := (-\infty,+\infty)$. The parameters p and q in this paper always satisfy $0 < q < p \leq 1$. The function $\theta^{-1}(\omega) = \Theta(\omega)$ as in (1.3) and $\mathsf{B}(\varkappa,\zeta)$ as in (1.5) are employed throughout this paper unless oterwise noted.

Definition 1.3. A function $\theta \in \Sigma$ is said to be the member of the family $\mathfrak{E}^{\delta,k}_{\Sigma,p,q}(x,\nu,\mu)$, if

$$\frac{1}{2}\left\{\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}+\left(\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}+\left(\frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\omega),$$

where $\zeta,\omega \in \mathfrak{D}$, $\varkappa \in (-\frac{1}{2},1]$, $k\in\mathbb{N}, 0<\delta\leq 1, \mu\geq 0$, and $\nu\in\mathbb{R}$ satisfying $\nu+\mu>0$.

The family $\mathfrak{E}^{\delta,k}_{\Sigma,p,q}(\varkappa,\nu,\mu)$ contains numerous new subclasses of Σ for specific chioces of $\delta,\nu,p,$ and q as listed below:

Example 1.1. $\mathfrak{F}^{\delta,k}_{\Sigma,p,q}(\varkappa,\mu)\equiv\mathfrak{E}^{\delta,k}_{\Sigma,p,q}(\varkappa,1-\mu,\mu), \varkappa\in(-\frac{1}{2},1], k\in\mathbb{N}, 0<\delta\leq 1$, and $\mu\geq 0$ is the set of functions θ in Σ that meet

$$\frac{1}{2}\left\{\frac{\zeta(A^{\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}+\left(\frac{\zeta(A^{\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{\omega(A^{\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}+\left(\frac{\omega(A^{\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\omega),$$

where $\zeta,\omega\in\mathfrak{D}$.

Example 1.2. $\mathfrak{G}^{\delta,k}_{\Sigma,p,q}(\varkappa,l,\mu)\equiv\mathfrak{E}^{\delta,k}_{\Sigma,p,q}(\varkappa,l-\mu+1,\mu), \varkappa\in(-\frac{1}{2},1], k\in\mathbb{N}$, $0<\delta\leq 1, l>-1$, and $\mu\geq 0$ is the set of $\theta\in\Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(C^{l,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}+\left(\frac{\zeta(C^{l,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{\omega(C^{l,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}+\left(\frac{\omega(C^{l,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\omega),$$

where $\zeta,\omega\in\mathfrak{D}$.

Example 1.3. $\mathfrak{H}^{k}_{\Sigma,p,q}(x,\nu,\mu)\equiv\mathfrak{E}^{1,k}_{\Sigma,p,q}(x,\nu,\mu)$ is the set of $\theta\in\Sigma$ functions that fulfill

$$\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}\prec \mathsf{B}(\varkappa,\zeta),\ \ \textit{and}\ \ \frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}\prec \mathsf{B}(\varkappa,\omega),\ \zeta,\omega\in\mathfrak{D},$$

where $\varkappa \in (-\frac{1}{2}, 1]$, $k \in \mathbb{N}, \mu \geq 0$, and $\nu \in \mathbb{R}$ such that $\nu + \mu > 0$.

Example 1.4. Given the set $\mathfrak{E}^{\delta,k}_{\Sigma,p,q}(x,\nu,\mu)$, if $q \to 1^-, p = 1$, we obtain $\mathfrak{K}^{\delta,k}_{\Sigma}(x,\nu,\mu)$, which is a collection of $\theta \in \Sigma$ functions that fulfill

$$\frac{1}{2}\left\{\frac{\zeta(\mathfrak{W}^{\nu,\mu.k}\theta(\zeta))'}{\theta(\zeta)} + \left(\frac{\zeta(\mathfrak{W}^{\nu,\mu.k}\theta(\zeta))'}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{\omega(\mathfrak{W}^{\nu,\mu.k}\Theta(\omega))'}{\Theta(\omega)} + \left(\frac{\omega(\mathfrak{W}^{\nu,\mu.k}\Theta(\omega))'}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\omega),$$

where $\mathfrak{W}^{\nu,\mu.k} \equiv W^{\nu,\mu,k}_{p=1,q\to 1^-}$, $k \in \mathbb{N}, 0 < \delta \leq 1$, $\mu \geq 0, \nu \in \mathbb{R}$ satisfying $\nu+\mu > 0$, $\varkappa \in (-\frac{1}{2}, 1]$, and $\zeta, \omega \in \mathfrak{D}$.

Section 2 contains estimates for $|d_2|$ and $|d_3|$ for functions $\in \mathfrak{S}^{\tau}_{\Sigma,p,q}(\varkappa,\nu,\mu)$ as well as Fekete-Szegö inequality [21]. Along with relevant connections to the earlier findings, there are also some fascinating implications of the primary finding.

2 Primary Findings

First, we determine the limits for $|d_2|$, $|d_3|$, and a Fekete-Szegö inequality for the members in $\mathfrak{S}^{\delta,k}_{\Sigma,p,q}(x,\nu,\mu)$.

Theorem 2.1. *Let $k \in \mathbb{N}$, $0 < \delta \leq 1, \mu \geq 0, \xi \in \mathbb{R}, \nu \in \mathbb{R}$ with $\nu + \mu > 0$, and $\varkappa \in (-\frac{1}{2}, 1]$. If a function $\theta \in \mathfrak{E}^{\delta,k}_{\Sigma,p,q}(x,\nu,\mu)$, then*

$$i). \quad |d_2| \leq \frac{6\delta\varkappa\sqrt{3\varkappa}}{\sqrt{|((1-\delta)\mathcal{M}^2 + (\mathcal{N}-\mathcal{M})2\delta(\delta+1))9\varkappa^2 - (1+\delta)^2\mathcal{M}^2(18\varkappa^2-1)|}}, \tag{2.1}$$

$$ii). \quad |d_3| \leq \frac{6\delta\varkappa}{\mathcal{N}(1+\delta)} + \frac{108\delta^2\varkappa^3}{|((1-\delta)\mathcal{M}^2 + (\mathcal{N}-\mathcal{M})2\delta(\delta+1))9\varkappa^2 - (1+\delta)^2\mathcal{M}^2(18\varkappa^2-1)|}, \tag{2.2}$$

and

$$iii). \quad |d_3 - \xi d_2^2| \leq \begin{cases} \frac{6\delta\varkappa}{(1+\delta)\mathcal{N}} & ; |1-\xi| \leq \mathcal{J} \\ \frac{108\delta^2\varkappa^3\,|1-\xi|}{|((1-\delta)\mathcal{M}^2+(\mathcal{N}-\mathcal{M})2\delta(\delta+1))9\varkappa^2-(1+\delta)^2\mathcal{M}^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{J}, \end{cases} \tag{2.3}$$

where

$$\mathcal{J} = \frac{|((1-\delta)\mathcal{M}^2 + (\mathcal{N}-\mathcal{M})2\delta(\delta+1))9\varkappa^2 - (1+\delta)^2\mathcal{M}^2(18\varkappa^2-1)|}{18\delta(1+\delta)\mathcal{N}\varkappa^2}, \tag{2.4}$$

$$\mathcal{M} = \left(2\left(\frac{\mu[2]_{p,q}+\nu}{\mu+\nu}\right)^k - 1\right), \tag{2.5}$$

and

$$\mathcal{N} = \left(3\left(\frac{\mu[3]_{p,q}+\nu}{\mu+\nu}\right)^k - 1\right). \tag{2.6}$$

Proof. Let $\theta \in \mathfrak{E}^{\delta,k}_{\Sigma,p,q}(x,\nu,\mu)$. Definition 1.3 can then be used to write

$$\frac{1}{2}\left\{\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)} + \left(\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\} = \mathsf{B}(\varkappa, m(\zeta)), \tag{2.7}$$

and

$$\frac{1}{2}\left\{\frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)} + \left(\frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\} = \mathsf{B}(\varkappa, n(\omega)), \tag{2.8}$$

where

$$\mathfrak{m}(\zeta) = m_1\zeta + m_2\zeta^2 + m_3\zeta^3 + ..., \text{ and } \quad \mathfrak{n}(\omega) = n_1\omega + n_2\omega^2 + n_3\omega^3 + ..., \zeta, \omega \in \mathfrak{D} \tag{2.9}$$

are some holomorphic functions with $|\mathfrak{m}(\zeta)| < 1$, $|\mathfrak{n}(\omega)| < 1$, ζ, $\omega \in \mathfrak{D}$ and $\mathfrak{m}(0) = 0 = \mathfrak{n}(0)$. We known that

$$|\mathfrak{m}_i| \leq 1, \text{ and } |\mathfrak{n}_i| \leq 1, \; i \in \mathbb{N}. \tag{2.10}$$

In light of (2.9), we obtain by substituting $\mathsf{B}(\varkappa,\zeta)$ from (1.5) in (2.7) and (2.8):

$$\mathsf{B}(\varkappa, m(\zeta)) = 1 + \mathsf{B}_1(\varkappa)m_1\zeta + [\mathsf{B}_1(\varkappa)m_2 + \mathsf{B}_2(\varkappa)m_1^2]\zeta^2 + ... \tag{2.11}$$

and

$$\mathsf{B}(\varkappa, n(\omega)) = 1 + \mathsf{B}_1(\varkappa)n_1\omega + [\mathsf{B}_1(\varkappa)n_2 + \mathsf{B}_2(\varkappa)n_1^2]\omega^2 + \dots. \tag{2.12}$$

The inference from (2.7) and (2.8) is that

$$\frac{1}{2}\left\{\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)} + \left(\frac{\zeta(W^{\nu,\mu.k}_{p,q}\theta(\zeta))'}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\} = \tag{2.13}$$

$$1 + \left(\frac{1+\delta}{2\delta}\right)\mathcal{M}d_2\zeta + \left(\left(\frac{1+\delta}{2\delta}\right)(\mathcal{N}d_3 - \mathcal{M}d_2^2) + \left(\frac{1-\delta}{4\delta^2}\right)\mathcal{M}^2d_2^2\right)\zeta^2 + ...$$

and

$$\frac{1}{2}\left\{\frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)} + \left(\frac{\omega(W^{\nu,\mu.k}_{p,q}\Theta(\omega))'}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\} = \tag{2.14}$$

$$1 + \left(\frac{1+\delta}{2\delta}\right)\mathcal{M}d_2\omega + \left(\left(\frac{1+\delta}{2\delta}\right)(\mathcal{N}(2d_2^2 - d_3) - \mathcal{M}d_2^2) + \left(\frac{1-\delta}{4\delta^2}\right)\mathcal{M}^2d_2^2\right)\omega^2 + \dots,$$

in which $\mathcal{M}$ and $\mathcal{N}$ are, respectively, as indicated in (2.5) and (2.6). Equations (2.13) and (2.14) imply that

$$\left(\frac{1+\delta}{2\delta}\right)\mathcal{M}d_2 = \mathsf{B}_1(\varkappa)\mathfrak{m}_1, \tag{2.15}$$

$$\left(\frac{1+\delta}{2\delta}\right)(\mathcal{N}d_3 - \mathcal{M}d_2^2) + \left(\frac{1-\delta}{4\delta^2}\right)\mathcal{M}^2 d_2^2 = \mathsf{B}_1(\varkappa)\mathfrak{m}_2 + \mathsf{B}_2(\varkappa)\mathfrak{m}_1^2, \tag{2.16}$$

$$-\left(\frac{1+\delta}{2\delta}\right)\mathcal{M}d_2 = \mathsf{B}_1(\varkappa)\mathfrak{n}_1 \tag{2.17}$$

and

$$\left(\frac{1+\delta}{2\delta}\right)(\mathcal{N}(2d_2^2 - d_3) - \mathcal{M}d_2^2) + \left(\frac{1-\delta}{4\delta^2}\right)\mathcal{M}^2 d_2^2 = \mathsf{B}_1(\varkappa)\mathfrak{n}_2 + \mathsf{B}_2(\varkappa)\mathfrak{n}_1^2. \tag{2.18}$$

From (2.15) and (2.17), we have

$$m_1 = -n_1 \tag{2.19}$$

and also

$$\left(\frac{(1+\delta)^2}{2\delta^2}\right)\mathcal{M}^2 d_2^2 = (m_1^2 + n_1^2)(\mathsf{B}_1(\varkappa))^2. \tag{2.20}$$

When (2.16) and (2.18) are added, we get

$$2\left[\left(\frac{1+\delta}{\delta}\right)(\mathcal{N}-\mathcal{M}) + \left(\frac{1-\delta}{2\delta^2}\right)\mathcal{M}^2\right] d_2^2 = \mathsf{B}_1(\varkappa)(m_2+n_2) + \mathsf{B}_2(\varkappa)(m_1^2 + sn_1^2). \tag{2.21}$$

Substituting the value of $m_1^2 + n_1^2$ from (2.20) in (2.21), we get

$$d_2^2 = \frac{2\delta^2(\mathsf{B}_1(\varkappa))^3(m_2+n_2)}{[((1-\delta)\mathcal{M}^2 + (\mathcal{N}-\mathcal{M})2\delta(\delta+1))(\mathsf{B}_1(\varkappa))^2 - (1+\delta)^2\mathcal{M}^2\mathsf{B}_2(\varkappa)]}, \tag{2.22}$$

which produces (2.1), when applied (2.10).

After deducting (2.18) from (2.16) and using (2.19), we arrive at

$$d_3 = d_2^2 + \frac{\delta\mathsf{B}_1(\varkappa)(m_2 - n_2)}{(1+\delta)\mathcal{N}}. \tag{2.23}$$

The inequality that results from this is as follows:

$$|d_3| \le |d_2|^2 + \frac{|\mathsf{B}_1(\varkappa)||\mathfrak{m}_2 - \mathfrak{n}_2|}{\left(\frac{\delta+1}{\delta}\right)U[3]_{p,q}}. \tag{2.24}$$

Applying (2.10) for $\mathfrak{m}_2$ and $\mathfrak{n}_2$, we obtain (2.5) from (2.1) and (2.24).

From (2.22) and (2.23), for $\xi \in \mathbb{R}$, we get in view of (1.4),

$$|d_3 - \xi d_2^2| = |L_1(x)|\left|\left(F(\xi,x) + \frac{\delta}{(1+\delta)\mathcal{N}}\right)m_2 + \left(F(\xi,x) - \frac{\delta}{(1+\delta)\mathcal{N}}\right)n_2\right|,$$

where

$$F(\xi, x) = \frac{2\delta^2(1-\xi)\mathsf{B}_1^2(\varkappa)}{\left[((1-\delta)\mathcal{M}^2 + (\mathcal{N}-\mathcal{M})2\delta(\delta+1))\mathsf{B}_1^2(\varkappa) - (1+\delta)^2\mathcal{M}^2\mathsf{B}_2(\varkappa)\right]}.$$

Clearly

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{2\delta|\mathsf{B}_1(\varkappa)|}{(1+\delta)\mathcal{N}} & ; 0 \leq |B(\xi, x)| \leq \frac{\delta}{\mathcal{N}(1+\delta)} \\ 2|\mathsf{B}_1(\varkappa)||F(\xi, x)| & ; |B(\xi, x)| \geq \frac{\delta}{\mathcal{N}(1+\delta)}, \end{cases}$$

with $\mathcal{J}$ as in (2.4), brings us to the conclusion (2.3). Thus, the proof is finished. □

Corollary 2.1. *Let $\nu = 1 - \mu$ in Theorem 2.1. Then for a function $\theta \in \mathfrak{F}^{\delta,k}_{\Sigma,p,q}(\varkappa, \mu)$, the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, are given by (2.1), (2.5), and (2.3), respectively, with $\mathcal{M} = \mathcal{M}_1 = 2\mu([2]_{p,q} - 1)^k + 1$, and $\mathcal{N} = \mathcal{N}_1 = 3\mu([3]_{p,q} - 1)^k + 2$. $\mathcal{M}$ and $\mathcal{N}$ must be replaced with $\mathcal{M}_1$ and $\mathcal{N}_1$ for $\mathcal{J}$ in (2.4).*

Corollary 2.2. *Let $\nu = 1 + l - \mu \, (l > -1)$, in Theorem 2.1. Then for $\theta \in \mathfrak{G}^{\delta,k}_{\Sigma,p,q}(\varkappa, l, \mu)$, the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, are given by (2.1), (2.5), and (2.3), respectively, with $\mathcal{M} = \mathcal{M}_2 = \left(2\left(\frac{l+1+\mu([2]_{p,q}-1)}{l+1}\right)^k - 1\right)$, and $\mathcal{N} = \mathcal{N}_2 = \left(3\left(\frac{l+1+\mu([3]_{p,q}-1)}{l+1}\right)^k - 1\right)$. $\mathcal{M}$ and $\mathcal{N}$ are to be replaced with $\mathcal{M}_2$ and $\mathcal{N}_2$ for $\mathcal{J}$ in (2.4).*

$\delta = 1$ Theorem 2.1 suggests

Corollary 2.3. *Let $\varkappa \in (-\frac{1}{2}, 1]$, $k \in \mathbb{N}, \xi \in \mathbb{R}$, $\mu \geq 0$, and $\nu \in \mathbb{R}$ with $\nu + \mu > 0$. If a function $\theta \in \mathfrak{H}^{k}_{\Sigma,p,q}(x, \nu, \mu)$, then*

$$i). \quad |d_2| \leq \frac{3\varkappa\sqrt{3\varkappa}}{\sqrt{|(\mathcal{N}-\mathcal{M})9\varkappa^2 - \mathcal{M}^2(18\varkappa^2 - 1)|}},$$

$$ii). \quad |d_3| \leq \frac{27\varkappa^2}{|(\mathcal{N}-\mathcal{M})9\varkappa^2 - \mathcal{M}^2(18\varkappa^2 - 1)} + \frac{3\varkappa}{\mathcal{N}}$$

and

$$iii). \quad |d_3 - \xi d_2^2| \leq \begin{cases} \frac{3\varkappa}{\mathcal{N}} & ; |1-\xi| \leq \mathcal{J}_1 \\ \frac{27x^3\,|1-\xi|}{|(\mathcal{N}-\mathcal{M})9\varkappa^2 - \mathcal{M}^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{J}_1, \end{cases}$$

where $\mathcal{J}_1 = \left|\frac{(\mathcal{N}-\mathcal{M})9\varkappa^2 - \mathcal{M}^2(18\varkappa^2 - 1)}{9\mathcal{N}\varkappa^2}\right|$, $\mathcal{M}$ and $\mathcal{N}$ are given by (2.5) and (2.6), respectively.

Remark 2.1. The result of Hussen and Illafe [27, Corollary 1] is obtained by allowing $k = 0$ in the above corollary.

Corollary 2.4. *In Theorem 2.1, let $q \to 1^-$ and $p = 1$. Then for any function $\theta \in \mathfrak{Y}^{\delta,k}_{\Sigma}(\varkappa, \nu, \mu)$, the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, are given by (2.1), (2.5), and (2.3), respectively, with $\mathcal{M} = \mathcal{M}_3 = \left(2\left(\frac{\nu+2\mu}{\nu+\mu}\right)^k - 1\right)$, and $\mathcal{N} = \mathcal{N}_3 = \left(3\left(\frac{\nu+3\mu}{\nu+\mu}\right)^k - 1\right)$. For J in (2.4), $\mathcal{M}$ and $\mathcal{N}$ are to be substituted with $\mathcal{M}_3$ and $\mathcal{N}_3$, respectively.*

Remark 2.2. $k = 0$ in the set $\mathfrak{Y}_{\Sigma}^{\delta,k}(\varkappa,\nu,\mu)$ yields the subset $\mathfrak{Q}_{\Sigma}^{\delta}(\varkappa)$, $0 < \delta \leq 1$, and $\varkappa \in (-\frac{1}{2}, 1]$ which is a collection of $\theta \in \Sigma$ functions that fulfill

$$\frac{1}{2}\left\{\frac{\zeta\theta'(\zeta)}{\theta(\zeta)} + \left(\frac{\zeta\theta'(\zeta)}{\theta(\zeta)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\zeta) \text{ and } \frac{1}{2}\left\{\frac{\omega\Theta'(\omega)}{\Theta(\omega)} + \left(\frac{\omega\Theta'(\omega)}{\Theta(\omega)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\omega),$$

where $\zeta, \omega \in \mathfrak{D}$.

Corollary 2.5. *Let* $0 < \delta \leq 1$ *and* $\varkappa \in (-\frac{1}{2}, 1]$. *If a function* $\theta \in \mathfrak{Q}_{\Sigma}^{\delta}(\varkappa)$, *then*

$$i). \quad |d_2| \leq \frac{6\delta\varkappa\sqrt{3\varkappa}}{\sqrt{|(1+\delta)^2 - (1+3\delta)9\varkappa^2)|}},$$

$$ii). \quad |d_3| \leq \frac{108\delta^2\varkappa^3}{|(1+\delta)^2 - (1+3\delta)9\varkappa^2)|} + \frac{3\delta\varkappa}{(1+\delta)},$$

and for $\xi \in \mathbb{R}$

$$iii). \quad |d_3 - \xi d_2^2| \leq \begin{cases} \frac{3\delta\varkappa}{(1+\delta)} & ; |1-\xi| \leq \frac{|(1+\delta)^2-(1+3\delta)9\varkappa^2)|}{36\delta(1+\delta)\varkappa^2} \\ \frac{108\delta^2\varkappa^3\,|1-\xi|}{|(1+\delta)^2-(1+3\delta)9\varkappa^2)|} & ; |1-\xi| \geq \frac{|(1+\delta)^2-(1+3\delta)9\varkappa^2)|}{36\delta(1+\delta)\varkappa^2}. \end{cases}$$

Remark 2.3. We derive the result of Hussen and Illafe [27, Corollary 1] by taking $\delta = 1$ in Corollary 2.5.

3 Conclusions

Upper bounds on $|d_2|$ and $|d_3|$ for funcions in the defined subfamily of σ associated with the (p,q)-derivative operator subordinate to LB polynomials are established in this study. Furthermore, for functions in this subfamily, the Fekete-Szegö functional $|d_3 - \mu d_2^2|, \mu \in \mathbb{R}$ has been noted. There have been few implications revealed by choosing the parameters in Theorem 2.1. Additionally, pertinent links to the ongoing research are found. However, not all of the important subclasses of Σ that are present in the literature are covered in this paper. For example, authors [33, 34, 40] have studied a number of subclasses of Σ involving the (p,q)-operator. The interested reader is advised to read these papers and the related references. Future studies might look into extending obtained results to fractional derivatives, Toeplitz determinants or higher-order Hankel determinants.

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