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Interpolative Berinde Weak Cyclic Contraction Mapping Principle

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Abstract

In this paper we introduce the notion of an interpolative Berinde weak cyclic operator. Additionally, we prove the existence and uniqueness of fixed point for such operators in metric space.

1 Introduction and Preliminaries

Definition 1.1. *[1] Let (X, d) be a metric space, and let A and B be two nonempty subsets of X. A mapping $T : A \cup B \mapsto A \cup B$ is said to be a cyclic mapping provided that $T(A) \subseteq B$ and $T(B) \subseteq A$.*

Definition 1.2. *[2] Let (X, d) be a metric space, and let A and B be two nonempty subsets of X. A point $x \in A \cup B$ of the map $T : A \cup B \mapsto A \cup B$ is a best proximity point if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.*

Definition 1.3. *[3] Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T : A \cup B \mapsto A \cup B$ is said to be a Kannan type cyclic contraction if there exists $k \in (0, \frac{1}{2})$ such that $d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y))$ for all $x \in A$ and all $y \in B$.*

Theorem 1.4. *[3] Let A and B be nonempty closed subsets of a complete metric space (X, d), and let $T : A \cup B \mapsto A \cup B$ be a Kannan type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Definition 1.5. *[4] Let (X, d) be a metric space, and let A and B be nonempty subsets of X. A cyclic map $T : A \cup B \mapsto A \cup B$ is said to be an interploative Kannan type cyclic contraction if there exists $k \in [0, 1)$ and $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq k[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha}$ for all $(x, y) \in A \times B$ with $x, y \notin Fix(T)$.*

Theorem 1.6. *[4] Let (X, d) be a complete metric space, and let A and B be nonempty subsets of X, and let $T : A \cup B \mapsto A \cup B$ be an interpolative Kannan type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

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Definition 1.7. *[4] Let (X, d) be a metric space and let A and B be nonempty subsets of X. A cyclic map $T : A \cup B \mapsto A \cup B$ is said to be an interpolative Reich-Rus-Ciric type cyclic contraction if there exists $k \in [0, 1)$ and positive reals α and β with $\alpha+\beta < 1$ such that $d(Tx, Ty) \leq k[d(x, y)]^{\beta}[d(Tx, x)]^{\alpha}[d(Ty, y)]^{1-\alpha-\beta}$ for all $(x, y) \in A \times B$ with $x, y \notin Fix(T)$.*

Theorem 1.8. *[4] Let (X, d) be a complete metric space and let A and B be nonempty subsets of X, and let $T : A \cup B \mapsto A \cup B$ be an interpolative Reich-Rus-Ciric type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Definition 1.9. *[5] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{1-\alpha}$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows

Definition 1.10. *[5] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx, Ty) \leq \lambda d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}},$$

where $\lambda \in [0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Theorem 1.11. *[5] Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator. If (X, d) is complete, then the fixed point of T exists.*

2 Main Result

Definition 2.1. *Let (X, d) be a metric space, and let A and B be nonempty subsets of X. A cyclic map $T : A \cup B \mapsto A \cup B$ will be called an interpolative Berinde weak type cyclic contraction if there exists $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{1-\alpha}$ for all $(x, y) \in A \times B$ with $x, y \notin Fix(T)$.*

Theorem 2.2. *Let (X, d) be a complete metric space, and let A and B be nonempty subsets of X. Suppose $T : A \cup B \mapsto A \cup B$ is an interpolative Berinde weak type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Proof. Fix $x \in A$. From Definition 2.1, we have

$$d(T^2x, Tx) \leq \lambda d(Tx, x)^{\alpha} d(Tx, T^2x)^{1-\alpha}$$

from which it follows that

$$d(T^2x, Tx) \leq td(Tx, x), \text{ where } t = \lambda^{\frac{1}{\alpha}}.$$

Thus we have $d(T^{n+1}x, T^n x) \leq t^n d(Tx, x)$. Consequently,

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^n x) \leq \left(\sum_{n=1}^{\infty} t^n\right) d(Tx, x) < \infty.$$

Hence $\{T^n x\}$ is a Cauchy sequence. It follows that there exists $z \in A \cup B$ such that $T^n x \to z$. Notice that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n+1}x\}$ is a sequence in B having the same limit z. As A and B are closed we conclude $z \in A \cap B$, that is, $A \cap B$ is nonempty. We claim that $Tz = z$. Observe that

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} d(Tz, T^{2n}x) \\ &= \lim_{n\to\infty} d(Tz, TT^{2n-1}x) \\ &\leq \lim_{n\to\infty} \lambda d(z, T^{2n-1}x)^{\alpha} d(z, Tz)^{1-\alpha} \\ &= 0. \end{aligned}$$

It follows that $d(z, Tz) = 0$, that is, $Tz = z$. To prove the uniqueness of the fixed point z, assume that there exists $w \in A \cap B$ such that $z \neq w$ and $Tw = w$. Taking into account that T is cyclic we get $w \in A \cap B$. Now we have

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &\leq \lambda d(z, w)^{\alpha} d(z, Tz)^{1-\alpha} \\ &= \lambda d(z, w)^{\alpha} d(z, z)^{1-\alpha} \\ &= 0. \end{aligned}$$

It follows that $d(z, w) = 0$, that is, $z = w$, and hence z is the unique fixed point of T. □

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