Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 2, 2025, Pages 211-234
https://doi.org/10.34198/ejms.15225.211234

A New Product for Soft Sets with its Decision-Making: Soft Gamma-Product

Aslıhan Sezgin[1], Eylül Şenyiğit[2] and Murat Luzum[3]

[1] Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye

[2] Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye

[3] Department of Mathematics, Faculty of Science, Van Yüzüncü Yıl University, Van, Türkiye

Abstract

Soft sets provide a strong mathematical foundation for managing uncertainty and give creative answers to parametric data challenges. In soft set theory, soft set operations are essential components. The "soft gamma-product," a novel product operation for soft sets, is presented in this study along with a detailed analysis of its algebraic features with respect to different kinds of soft equalities and subsets. We further explore the soft gamma-product's relation with other soft set operations by examining its distributions over other soft set activities. Using the *uni-int* operator and *uni-int* decision function within the soft gamma-product for the *uni-int* decision-making approach, which finds an ideal collection of components from accessible possibilities, we end with an example showing the method's efficacy of many applications. Since the theoretical underpinnings of soft computing techniques are based on sound mathematical concepts, this study makes a substantial contribution to the literature on soft sets.

1. Introduction

Modern set theory, the foundation of all mathematics, was created by George Cantor. One problem with the concept of a set is ambiguity, as mathematics requires correctness in all conceptions, including sets. This ambiguity or depiction of partial information has long been a source of difficulty for mathematicians, logicians, and philosophers. It has also been a greater worry for computer scientists in recent years, particularly in the area of artificial intelligence. For modeling complex systems, a variety of mathematical approaches are available, such as interval mathematics, fuzzy set theory [1], and probability theory; nevertheless, each has a unique set of limitations. Probability theory only works with stochastically stable systems, interval mathematics suffers from fluctuating uncertainty, and fuzzy set theory is known to have issues with setting membership values. Furthermore, these techniques' lack of parameterization limits their effectiveness, especially in complex domains like the social sciences, environmental science, and economics. Soft set theory was introduced by Russian researcher Molodtsov [2] in 1999 as a fully generic mathematical method for characterizing uncertainty. Researchers can change parameters as needed because item descriptions are not strictly restricted. This greatly simplifies and enhances decision-making, especially when information is scarce. By overcoming the obstacles and offering a greater variety of applications in various areas, soft set theory sets itself apart.

A soft set gives a rough description of an object and is composed of a predicate and an estimated value set. Approximation techniques are used for complex models without accurate solutions, even if models in traditional mathematics demand precise answers. Conversely, since the initial description of an item is inherently

Received: December 8, 2024; Revised: December 29, 2024; Accepted: December 31, 2024; Published: January 7, 2025

2020 Mathematics Subject Classification: 03E20, 03E72.

Keywords and phrases: soft set, soft gamma-product, soft subset, soft equal relation.

*Corresponding author

imprecise, soft set theory does not necessitate a precise solution notion. By successfully applying soft set theory to a number of domains, including Riemann integration, operations research, game theory, and function smoothness, Molodtsov [2] showed the theory's adaptability. Several scholars [4-10] developed early soft set-based decision-making strategies after Maji et al. [3] introduced soft set theory to a decision-making problem for the first time. The well-known soft set-based strategy known as the "*uni-int* decision-making" method was initially published by Çağman and Enginoğlu [11]. For the OR, AND, AND-NOT, and OR-NOT operations, soft matrix-based decision-making methods were later introduced [12]. Since these methods have been proven to be effective in handling uncertainty and other real-world problems, soft set theory has been widely applied in decision-making [13-24].

Over the past few years, there has been a substantial advancement in the foundations of soft set theory. Maji et al. [25] offered a comprehensive theoretical framework that included soft subsets, soft set equality, and soft set operations such as union, intersection, and AND/OR products. Pei and Miao [26] expanded on these concepts by redefining intersection and subset relations and examining connections to information systems. Ali et al. [27] provided additional operations, including restricted union, restricted intersection, restricted difference, and extended intersection. Subsequent publications [28-41] explored the algebraic structure of soft sets, proposed improvements, and addressed conceptual inconsistencies in previous soft set research. Eren and Çalışıcı [42] developed a new type of difference operation for soft sets, while Stojanovic [43] studied the extended symmetric difference of soft sets. Other new soft set techniques have already been developed and studied [44-49]. Subsets and soft equality relations are fundamental concepts in soft set theory. Maji et al. [25] introduced the idea of soft subsets, which Pei and Miao [26] and Feng et al. [29] further developed. Qin and Hong [50] introduced two new types of congruence relations and soft equal relations on soft sets. By employing a greater range of soft subsets, Jun and Yang [51] expanded soft equal relations and amended Maji's soft distributive laws. This work created J-soft equal relations for consistency. These developments led Liu, Feng, and Jun [52] to propose soft L-subsets and soft L-equal relations, pointing out that not all soft equalities follow distributive norms.

Building on earlier research, Feng et al. [53] investigated the algebraic features of soft product operations, including laws of distribution, commutativity, and association, among other characteristics, and expanded on the categories of soft subsets. They examined soft products such as AND and OR products using soft L-subsets, examining these operations under J-equality and L-equality. Additionally, they demonstrated that soft L-equal relations may coexist with commutative semigroup structures. For further details on soft equal relations, see [54-58]. Çağman and Enginoğlu [11] refined Molodtsov's original concept of soft sets and introduced several products in soft set theory, including *uni-int* decision functions, AND-products, OR-products, AND-NOT-products, and OR-NOT-products. They provided a practical example of how this approach may handle ambiguity by using these things to develop a methodical decision-making process for selecting the best options among alternatives. Sezgin et al. [59] conducted a comprehensive analysis of the AND-product, examining its algebraic properties (idempotent, commutative, and associative laws) and contrasting them with those of several soft equalities, including soft F, M, L, and J equalities. They demonstrated that the collection of all soft sets over the universe forms a commutative hemiring with identity under soft L-equality when paired with the restricted/extended union and the AND-product. Furthermore, they showed that this structure also holds when combining the restricted/extended symmetric difference with the AND-product, establishing another commutative hemiring with identity in the context of soft L-equality.

A new product operation in soft set theory is introduced in this paper, which we call the "soft gamma-product." Concerning different types of equality and soft subsets, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality, we evaluate the algebraic properties of this operation and provide an example. We also examine the distributional properties of this product in various soft set operations. Finally, we demonstrate the effectiveness of the soft decision-making approach in using it to select the best options in a decision-making scenario. This paper contributes to the corpus of literature on soft sets by creating the theoretical foundations required for applications in soft computing. The paper is organized as follows: Section 2 provides a summary of the key concepts in soft set theory. We introduce the soft gamma-product and discuss its algebraic properties concerning several soft equalities and subsets in the third part. Section 4 looks at the application of soft gamma-product and *uni-int* decision operators in decision-making. Concluding remarks are included in the final section.

2. Preliminaries

Definition 2.1. [1] Let U be the universal set, E be the parameter, and P(U) be the power set of U and $\mathcal{K} \subseteq \mathrm{E}$. A pair $(\mathfrak{J}, \mathcal{K})$ is called a soft set over U, where $\mathfrak{J}$ is a set-valued function such that $\mathfrak{J}: \mathcal{K} \to \mathrm{P(U)}$.

Although Çağman and Enginoğlu [11] modified Molodstov's concept of soft sets, we continue to use the original definition of soft set in our work. Throughout this paper, the collection of all the soft sets defined over U is designated as $\mathrm{S_E(U)}$. Let $\mathcal{K}$ be a fixed subset of E and $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$ be the collection of all those soft sets over U with the fixed parameter set $\mathcal{K}$. That is, while in the set $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$, there are only soft sets whose parameter sets are $\mathcal{K}$; in the set $\mathrm{S_E(U)}$, there are soft sets whose parameter sets may be any set. From now on, while soft set will be designated by SS and parameter set by PS; soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

Definition 2.2. [27] Let $(\mathfrak{J}, \mathcal{K})$ be an SS over U. $(\mathfrak{J}, \mathcal{K})$ is called a relative null SS (with respect to the PS $\mathcal{K}$), denoted by $\emptyset_{\mathcal{K}}$, if $\mathfrak{J}(\mathscr{k}) = \emptyset$ for all $\mathscr{k} \in \mathcal{K}$ and $(\mathfrak{J}, \mathcal{K})$ is called a relative whole SS (with respect to the PS $\mathcal{K}$), denoted by $\mathrm{U}_{\mathcal{K}}$ if $\mathfrak{J}(\mathscr{k}) = \mathrm{U}$ for all $\mathscr{k} \in \mathcal{K}$. The relative whole SS $\mathrm{U_E}$ with respect to the universe set of parameters E is called the absolute SS over U.

The empty SS over U is the unique SS over U with an empty PS, represented by $\emptyset_\emptyset$. Note $\emptyset_\emptyset$ and $\emptyset_{\mathcal{M}}$ are different [31]. In the following, we always consider SSs with non-empty PSs in the universe U, unless otherwise stated.

The concept of soft subset, which we refer to here as soft M-subset to prevent confusion, was initially defined by Maji et al. [25] in the following extremely strict way:

Definition 2.3. [25] Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be two SSs over U. $(\mathfrak{J}, \mathcal{K})$ is called a soft M-subset of $(\mathfrak{S}, \mathcal{Z})$ denoted by $(\mathfrak{J}, \mathcal{K}) \,\tilde{\subseteq}_{\mathrm{M}}\, (\mathfrak{S}, \mathcal{Z})$ if $\mathcal{K} \subseteq \mathcal{Z}$ and $\mathfrak{J}(\mathscr{k}) = \mathfrak{S}(\mathscr{k})$ for all $\mathscr{k} \in \mathcal{K}$. Two SSs $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ are said to be soft M-equal, denoted by $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} (\mathfrak{S}, \mathcal{Z})$ if $(\mathfrak{J}, \mathcal{K}) \,\tilde{\subseteq}_{\mathrm{M}}\, (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z}) \,\tilde{\subseteq}_{\mathrm{M}}\, (\mathfrak{J}, \mathcal{K})$.

Definition 2.4. [26] Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be two SSs over U. $(\mathfrak{J}, \mathcal{K})$ is called a soft F-subset of $(\mathfrak{S}, \mathcal{Z})$ denoted by $(\mathfrak{J}, \mathcal{K}) \,\tilde{\subseteq}_{\mathrm{F}}\, (\mathfrak{S}, \mathcal{Z})$ if $\mathcal{K} \subseteq \mathcal{Z}$ and $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(\mathscr{k})$ for all $\mathscr{k} \in \mathcal{K}$. Two SSs $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ are said to be soft F-equal, denoted by $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{F}} (\mathfrak{S}, \mathcal{Z})$ if $(\mathfrak{J}, \mathcal{K}) \,\tilde{\subseteq}_{\mathrm{F}}\, (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z}) \,\tilde{\subseteq}_{\mathrm{F}}\, (\mathfrak{J}, \mathcal{K})$.

It is important to note that the definitions of soft F-subset and soft F-equal were originally introduced by Pei and Miao in [26]. However, some papers on soft subsets and soft equalities mistakenly attribute these definitions to Feng et al. [29]. Consequently, the letter "F" is used to reference this connection.

In [52], it was shown that the soft equality relations =M and =F are equivalent. In other words, $((\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if and only if $(\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$. Since they have the same set of parameters and approximation function, two SSs that satisfy this equivalence are actually identical [52], meaning that $(\mho,\mathcal{M}) =_{\mathrm{M}} (\mathfrak{F},\mathcal{D})$ implies $(\mho,\mathcal{M}) = (\mathfrak{F},\mathcal{D})$.

Jun and Yang [51] expanded the concepts of F-soft subsets and soft F-equal relations by relaxing the restrictions on parameter sets (PSs). Although they referred to these as the generalized soft subset and generalized soft equal relation in [51], we refer to them as soft J-subsets and soft J-equal relations, taking the initial letter of Jun.

Definition 2.5. [51] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. $(\mathfrak{J},\mathcal{K})$ is called a soft J-subset of $(\mathfrak{S},\mathcal{Z})$ denoted by $(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ if for all $\hbar\in\mathcal{K}$, there exists $z \in \mathcal{Z}$ such that $\mathfrak{J}(\hbar) \subseteq \mathfrak{S}(z)$. Two SSs $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ are said to be soft J-equal, denoted by $(\mathfrak{J},\mathcal{K}) =_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ if $(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{J},\mathcal{K})$.

In [52] and [53], it was demonstrated that $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F},\mathcal{D})$, but the converse may not be true.

Liu, Feng, and Jun [52] introduced a new type of soft subset, referred to as soft L-subsets and soft L-equality, which generalizes both soft M-subsets and ontology-based soft subsets. This new concept was inspired by the ideas of soft J-subsets [51] and ontology-based soft subsets [30].

Definition 2.6. [52] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. $(\mathfrak{J},\mathcal{K})$ is called a soft L-subset of $(\mathfrak{S},\mathcal{Z})$ denoted by $(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{S},\mathcal{Z})$ if for all $\hbar\in\mathcal{K}$, there exists $z \in \mathcal{Z}$ such that $\mathfrak{J}(\hbar) = \mathfrak{S}(z)$. Two SSs $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ are said to be soft J-equal, denoted by $(\mathfrak{J},\mathcal{K}) =_{\mathrm{L}} (\mathfrak{S},\mathcal{Z})$ if $(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{J},\mathcal{K})$.

Concerning the relationships among various types of soft subsets and soft equalities, $(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{J},\mathcal{K}) =_{\mathrm{L}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{J},\mathcal{K}) =_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ [52]. However, the converses may not be true. Also, it is well-known that $(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} (\mathfrak{S},\mathcal{Z})$ if and only if $(\mathfrak{J},\mathcal{K}) =_{\mathrm{F}} (\mathfrak{S},\mathcal{Z})$.

We can thus conclude that soft M-equality (and therefore soft F-equality) represents the strictest form of soft equality, while soft J-equality is the weakest. Positioned between these two is the concept of soft L-equality [52].

For further information on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and other definitions of soft subsets and soft equal relations in the literature, please refer to [50-58].

Definition 2.7. [27] Let $(\mathfrak{J},\mathcal{K})$ be an SS over U. The relative complement of $(\mathfrak{J},\mathcal{K})$, denoted by $(\mathfrak{J},\mathcal{K})^{\mathrm{r}}$, is defined by $(\mathfrak{J},\mathcal{K})^{\mathrm{r}} = (\mathfrak{J}^{r},\mathcal{K})$, where $\mathfrak{J}^{\mathrm{r}}\colon \mathcal{K} \to \mathrm{P(U)}$ is a mapping given by $\mathfrak{J}^{\mathrm{r}}(\hbar) = \mathrm{U}\backslash\mathfrak{J}(\hbar)$ for all $\hbar \in \mathcal{K}$. From now on, $\mathrm{U}\backslash\mathfrak{J}(\hbar)=[\mathfrak{J}(\hbar)]'$ is designated by $\mathfrak{J}'(\hbar)$ for the sake of designation.

Definition 2.8. [25] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. The AND-product ($\wedge$-product) of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K})\wedge(\mathfrak{S},\mathcal{Z})$, is defined by $(\mathfrak{J},\mathcal{K})\wedge(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})$, where for all $(\hbar,z) \in \mathcal{K}\mathrm{x}\mathcal{Z}$, $\mathfrak{Q}(\hbar,z) = \mathfrak{J}(\hbar) \cap \mathfrak{S}(z)$.

Definition 2.9. [25] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. The OR-product ($\vee$-product) of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \vee (\mathfrak{S},\mathcal{Z})$, and is defined by $(\mathfrak{J},\mathcal{K}) \vee (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})$, where for all $(\hbar,z) \in \mathcal{K}\mathrm{x}\mathcal{Z}$, $\mathfrak{Q}(\hbar,z) = \mathfrak{J}(\hbar) \cup \mathfrak{S}(z)$.

Çağman [60] introduced the concepts of inclusive complement and exclusive complement as novel ideas in set theory, examining the relationships between them through comparison. These new concepts were also applied to group theory in [60]. Sezgin et al. [61] introduced some new complements and investigated the relations between them and applied them to group theory as well.

Definition 2.10. [61] Let A and B be two subsets of the universe. Then, A gamma B is defined by $A\gamma B := A' \cap B$.

Subsequently, the gamma operation was applied to SS theory to introduce new SS operations [62-65]. Let "$\odot$" represent set operations such as $\cap$, $\cup$, $\setminus$, $\triangle$. The following definitions are provided for restricted, extended, and soft binary piecewise operations.

Definition 2.11. [27] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The restricted $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \odot_R (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K}) \odot_R (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\cap\mathcal{Z})$, where $\mathcal{C} = \mathcal{K} \cap \mathcal{Z}$ and if $\mathcal{C} \neq \emptyset$, then for all $c \in \mathcal{C}$, $\mathfrak{Q}(c) = \mathfrak{J}(c) \odot \mathfrak{S}(c)$; if $\mathcal{C} = \emptyset$, then $(\mathfrak{J},\mathcal{K}) \odot_R (\mathfrak{S},\mathcal{Z}) = \emptyset_\emptyset$.

Definition 2.12. [27,43,62] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The extended $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \odot_\varepsilon (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K}) \odot_\varepsilon (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\cup\mathcal{Z})$, where $\mathcal{C} = \mathcal{K} \cup \mathcal{Z}$ and for all $c \in \mathcal{C}$,

$$\mathfrak{Q}(c) = \begin{cases} \mathfrak{J}(c), & c \in \mathcal{K}\setminus \mathcal{Z} \\ \mathfrak{S}(c), & c \in \mathcal{Z}\setminus\mathcal{K} \\ \mathfrak{J}(c) \odot \mathfrak{S}(c), & c \in \mathcal{K} \cap \mathcal{Z}. \end{cases}$$

Definition 2.13. [44,65] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The soft binary piecewise $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \widetilde{\odot} (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K}) \widetilde{\odot} (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K})$, where for all $c \in \mathcal{K}$,

$$\mathfrak{Q}(c) = \begin{cases} \mathfrak{J}(c), & c \in \mathcal{K}\setminus \mathcal{Z} \\ \mathfrak{J}(c) \odot \mathfrak{S}(c), & c \in \mathcal{K} \cap \mathcal{Z}. \end{cases}$$

For more about soft algebraic structures of SSs and picture fuzzy soft sets, we refer to [66-94].

3. Soft Gamma-Product and its Algebraic Properties

We proposed the soft gamma-product, a novel product for SSs, in this part. We provide an example and analyze its algebraic characteristics in depth with respect to specific kinds of soft equalities and soft subsets.

Definition 3.1. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The soft gamma-product of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$, is defined by $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})= (\mathfrak{Q},\mathcal{K}\text{x}\mathcal{Z})$, where for all $(\mathscr{k},z)\in \mathcal{K}\text{x}\mathcal{Z}$,

$$\mathfrak{Q}(\mathscr{k},z) = \mathfrak{J}(\mathscr{k})\gamma\mathfrak{S}(z).$$

Here, $\mathfrak{J}(\mathscr{k})\lambda\mathfrak{S}(z) = \mathfrak{J}'(\mathscr{k})\cap \mathfrak{S}(z)$.

Example 3.2. Let $E = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be the PS, $\mathcal{K} = \{\ell_2, \ell_3\}$, and $\mathcal{Z} = \{\ell_2, \ell_4\}$ be the subsets of E, $U = \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4, \mathfrak{k}_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U such that

$$(\mathfrak{J},\mathcal{K}) = \{(\ell_2, \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4, \mathfrak{k}_5\}), (\ell_3, \{\mathfrak{k}_3, \mathfrak{k}_5\})\},$$

$$(\mathfrak{S},\mathcal{Z}) = \{(\ell_2, \{\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3\})(\ell_4, \{\mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{k}_4\})\}.$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\text{x}\mathcal{Z})$. Thus,

$$(\mathfrak{Q},\mathcal{K}\text{x}\mathcal{Z}) = \{((\ell_2,\ell_2),\emptyset), ((\ell_2,\ell_4),\emptyset), ((\ell_3,\ell_2),\{\mathfrak{k}_1,\mathfrak{k}_2\}), ((\ell_3,\ell_4),\{\mathfrak{k}_2,\mathfrak{k}_4\})\}.$$

Since it is more practical than writing in the list method style, the table method can be applied here:

$(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\mathfrak{S},\mathcal{Z})$	ℓ_2	ℓ_4
ℓ_2	$\emptyset$	$\emptyset$
ℓ_3	$\{\mathfrak{k}_1,\mathfrak{k}_2\}$	$\{\mathfrak{k}_2,\mathfrak{k}_4\}$

Proposition 3.3. Λ_{γ}*-product is closed in* $\mathrm{S_E(U)}$.

Proof: It is clear that Λ_{γ}-product in a binary operation in $\mathrm{S_E(U)}$. In fact, let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then,

$$\Lambda_{\gamma}: \mathrm{S_E(U)} \times \mathrm{S_E(U)} \rightarrow \mathrm{S_E(U)}$$

$$\big((\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})\big) \rightarrow (\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z}) = (\mathfrak{Q},\mathcal{C}).$$

That is, $(\mathfrak{Q},\mathcal{C})$ is an SS over U, since the set $\mathrm{S_E(U)}$ contains all the SS over U. Here, note that the set $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$ is not closed under Λ_{γ}-product, since if $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{K})$ are the elements of $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$, $(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\mathfrak{S},\mathcal{K})$ is an element of $\mathrm{S}_{\mathcal{K}\mathrm{x}\mathcal{K}}(\mathrm{U})$, not $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$.

Proposition 3.4. *Let* $(\mathfrak{J},\mathcal{K})$,$(\mathfrak{S},\mathcal{Z})$ *and* $(\mathfrak{Q},\mathcal{C})$ *be SSs over* U. *Then*,

$$(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}\big[(\mathfrak{S},\mathcal{Z})\Lambda_{\gamma}(\mathfrak{Q},\mathcal{C})\big] \neq_{\mathrm{M}} \big[(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\mathfrak{S},\mathcal{Z})\big]\Lambda_{\gamma}(\mathfrak{Q},\mathcal{C}).$$

That is, Λ_{γ}*-product is not associative in* $\mathrm{S_E(U)}$.

Proof: We provided an example to show that V_{λ}-product is not associative in $\mathrm{S_E(U)}$. Let $\mathrm{E} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be PS, $\mathcal{K} = \{\ell_2, \ell_3\}$, $\mathcal{Z} = \{\ell_1\}$, and $\mathcal{C} = \{\ell_4\}$ be the subsets of E, $\mathrm{U} = \{\mathfrak{k}_1,\mathfrak{k}_2,\mathfrak{k}_3,\mathfrak{k}_4,\mathfrak{k}_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$,$(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U such that $(\mathfrak{J},\mathcal{K}) = \{(\ell_2,\{\mathfrak{k}_3,\mathfrak{k}_4\}),(\ell_3,\{\mathfrak{k}_1\})\}$, $(\mathfrak{S},\mathcal{Z}) = \{(\ell_1,\emptyset)\}$ and $(\mathfrak{Q},\mathcal{C}) = \{(\ell_4,\{\mathfrak{k}_1,\mathfrak{k}_3,\mathfrak{k}_5\})\}$. We show that

$$(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}\big[(\mathfrak{S},\mathcal{Z})\Lambda_{\gamma}(\mathfrak{Q},\mathcal{C})\big] \neq_{\mathrm{M}} \big[(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\mathfrak{S},\mathcal{Z})\big]\Lambda_{\gamma}(\mathfrak{Q},\mathcal{C}).$$

Let $(\mathfrak{S},\mathcal{Z})\Lambda_{\gamma}(\mathfrak{Q},\mathcal{C}) = (\zeta,\mathcal{Z}\times\mathrm{C})$. Thus,

$$(\zeta,\mathcal{Z}\times\mathrm{C}) = \big\{\big((\ell_1,\ell_4),\{\mathfrak{k}_1,\mathfrak{k}_3,\mathfrak{k}_5\}\big)\big\}.$$

Assume that $(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\zeta,\mathcal{Z}\times\mathrm{C}) = \big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big)$. Thereby,

$$\big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big) = \Big\{\Big(\big(\ell_2,(\ell_1,\ell_4)\big),\{\mathfrak{k}_1,\mathfrak{k}_5\}\Big),\Big(\big(\ell_3,(\ell_1,\ell_4)\big),(\mathfrak{k}_3,\mathfrak{k}_5)\Big)\Big\}.$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_{\gamma}(\mathfrak{S},\mathcal{Z}) = (\mathfrak{G},\mathcal{K}\times\mathcal{Z})$. Hence,

$$(\mathfrak{G},\mathcal{K}\times\mathcal{Z}) = \big\{\big((\ell_2,\ell_1),\emptyset\big),\big((\ell_3,\ell_1),\emptyset\big)\big\}.$$

Suppose that $(\mathfrak{G},\mathcal{K}\times\mathcal{Z})\Lambda_{\gamma}(\mathfrak{Q},\mathcal{C}) = (\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$. Therefore,

$$(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C}) = \Big\{\Big(\big((\ell_2,\ell_1),\ell_4\big),\{\mathfrak{k}_1,\mathfrak{k}_3,\mathfrak{k}_5\}\Big),\Big(\big((\ell_3,\ell_1),\ell_4\big),\{\mathfrak{k}_1,\mathfrak{k}_3,\mathfrak{k}_5\}\Big)\Big\}.$$

Thus, $(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times \mathrm{C}))\neq_{\mathrm{M}}(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$. Similarly, $(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times \mathrm{C}))\neq_{\mathrm{L}}(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$ and $(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times \mathrm{C}))\neq_{\mathrm{J}}(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$.

Proposition 3.5. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then*, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\neq_{\mathrm{M}}(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})$. *Namely*, Λ_γ*-product is not commutative in* $\mathrm{S_E(U)}$.

Proof: Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})=(\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})=(\mathfrak{G},\mathcal{Z}\times\mathcal{K})$. Since $\mathcal{K}\times\mathcal{Z}\neq\mathcal{Z}\times\mathcal{K}$, the rest of the proof is obvious.

Proposition 3.6. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. Then, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\neq_{\mathrm{J}}(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})$. *That is*, Λ_γ*-product is not commutative in* $\mathrm{S_E(U)}$ *under J-equality.*

Proof: We provided an example to show that Λ_γ-product is not commutative under J-equality in $\mathrm{S_E(U)}$. Let $\mathrm{E}=\{\ell_1,\ell_2,\ell_3,\ell_4\}$ be the PS, $\mathcal{K}=\{\ell_2,\ell_3\}$ and $\mathcal{Z}=\{\ell_1\}$ be the subsets of E, $\mathrm{U}=\{\mathfrak{k}_1,\mathfrak{k}_2,\mathfrak{k}_3,\mathfrak{k}_4,\mathfrak{k}_5\}$ be the universal set and $(\mathfrak{J},\mathcal{K})$ and$(\mathfrak{S},\mathcal{Z})$ be SSs over U such that $(\mathfrak{J},\mathcal{K})=\{(\ell_2,\{\mathfrak{k}_3,\mathfrak{k}_4\}),(\ell_3,\{\mathfrak{k}_1\})\}$, $(\mathfrak{S},\mathcal{Z})=\{(\ell_1,\emptyset)\}$. We show that

$$(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\neq_{\mathrm{J}}(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K}).$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})=(\mathcal{W},\mathcal{K}\times\mathcal{Z})$, where

$$(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})=(\mathcal{W},\mathfrak{J}\times\mathcal{Z})=\{((\ell_2,\ell_1),\emptyset),((\ell_3,\ell_1),\emptyset)\}.$$

Assume that $(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})=(\mathcal{H},\mathcal{Z}\times\mathcal{K})$, where

$$(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})=(\mathcal{H},\mathcal{Z}\times\mathcal{K})=\left(((\ell_1,\ell_2),\{\mathfrak{k}_3,\mathfrak{k}_4\})((\ell_1,\ell_3),\{\mathfrak{k}_1\})\right).$$

Hence, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\neq_{\mathrm{J}}(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})$. Moreover, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\neq_{\mathrm{L}}(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})$.

Proposition 3.7. *Let* $(\mathfrak{J},\mathcal{K})$ *be an SS over* U. *Then*, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma\emptyset_\emptyset=_{\mathrm{M}}\emptyset_\emptyset\Lambda_\gamma(\mathfrak{J},\mathcal{K})=_{\mathrm{M}}\emptyset_\emptyset$. *Namely*, $\emptyset_\emptyset$*-the empty SS-is the absorbing element of* Λ_γ*-product in* $\mathrm{S_E(U)}$.

Proof: Let $\emptyset_\emptyset=(\mathfrak{Q},\emptyset)$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma\emptyset_\emptyset=(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\emptyset)=(\mathfrak{S},\mathcal{K}\times\emptyset)=(\mathfrak{S},\emptyset)$. Since $\emptyset_\emptyset$ is the only SS whose PS is $\emptyset$, $(\mathfrak{S},\emptyset)=\emptyset_\emptyset$ is obtained. Similarly, $\emptyset_\emptyset\Lambda_\gamma(\mathfrak{J},\mathcal{K})=_{\mathrm{M}}\emptyset_\emptyset$.

Proposition 3.8. *Let* $(\mho,\mathcal{K})$ *be an SS over* U. *Then*, $(\mho,\mathcal{K})\Lambda_\gamma\emptyset_\mathcal{K}=_{\mathrm{L}}\emptyset_\mathcal{K}$. *That is*, $\emptyset_\mathcal{K}$ *is the right absorbing element of* Λ_γ*-product in* $\mathrm{S}_\mathcal{K}(\mathrm{U})$ *under L-equality.*

Proof: Let $\emptyset_\mathcal{K}=(\Upsilon,\mathcal{K})$ and $(\mho,\mathcal{K})\Lambda_\gamma(\Upsilon,\mathcal{K})=(\gtrless,\mathcal{K}\mathrm{x}\mathcal{K})$. Then, for all $k\in\mathcal{K}$, $\Upsilon(k)=\emptyset$ and for all $(k,z)\in\mathcal{K}\mathrm{x}\mathcal{K}$, $\gtrless(k,z)=\mho'(k)\cap\emptyset=\emptyset$. Since, for all $(k,z)\in\mathcal{K}\mathrm{x}\mathcal{K}$, there exists $k\in\mathcal{K}$ such that $\gtrless(k,z)=\emptyset=\Upsilon(k)$, $(\mho,\mathcal{K})\Lambda_\gamma\emptyset_\mathcal{K}\,\tilde{\subseteq}_{\mathrm{L}}\,\emptyset_\mathcal{K}$. Moreover, for all $k\in\mathcal{K}$, there exists $(k,z)\in\mathcal{K}\mathrm{x}\mathcal{K}$ such that $\Upsilon(k)=\emptyset=\gtrless(k,z)$, implying that $\emptyset_\mathcal{K}\,\tilde{\subseteq}_{\mathrm{L}}\,(\mho,\mathcal{M})\Lambda_\gamma\emptyset_\mathcal{K}$. Thereby, $\emptyset_\mathcal{K}\Lambda_\gamma(\mho,\mathcal{K})=_{\mathrm{L}}\emptyset_\mathcal{K}$.

Proposition 3.9. *Let* $(\mho,\mathcal{K})$ *be an SS over* U. *Then*, $\emptyset_\mathcal{K}\Lambda_\gamma(\mho,\mathcal{K})=_{\mathrm{L}}(\mho,\mathcal{K})$. *That is*, $\emptyset_\mathcal{K}$ *is the left identity element of* Λ_γ*-product in* $\mathrm{S}_\mathcal{K}(\mathrm{U})$ *under L-equality.*

Proof: Let $\emptyset_\mathcal{K}=(\Upsilon,\mathcal{K})$ and $(\Upsilon,\mathcal{K})\Lambda_\gamma(\mho,\mathcal{K})=(\gtrless,\mathcal{K}\mathrm{x}\mathcal{K})$. Then, for all $k\in\mathcal{K}$, $\Upsilon(k)=\emptyset$ and for all $(k,z)\in\mathcal{K}\mathrm{x}\mathcal{K}$, $\gtrless(k,z)=\Upsilon(k)\cap\mho(z)$=U $\cap\,\mho(z)=\mho(z)$. Since, for all $(k,z)\in\mathcal{M}\mathrm{x}\mathcal{K}$ there exists $z\in\mathcal{K}$ such that $\gtrless(k,z)=\mho(z)$, implying that $\emptyset_\mathcal{K}\Lambda_\gamma(\mho,\mathcal{K})\,\tilde{\subseteq}_{\mathrm{L}}\,(\mho,\mathcal{K})$. Moreover, for all $z\in\mathcal{K}$, there exists $(k,z)\in\mathcal{K}\mathrm{x}\mathcal{K}$ such that $\mho(z)=\gtrless(k,z)$, implying that $(\mho,\mathcal{K})\,\tilde{\subseteq}_{\mathrm{L}}\,\emptyset_\mathcal{K}\Lambda_\gamma(\mho,\mathcal{K})$ Thereby, $\emptyset_\mathcal{K}\Lambda_\gamma(\mho,\mathcal{K})$ $=_{\mathrm{L}}(\mho,\mathcal{K})$.

Proposition 3.10. *Let* $(\mathfrak{J},\mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma\emptyset_\mathcal{K} =_\mathrm{M} \emptyset_{\mathcal{K}\times\mathcal{K}}$ *and* $\emptyset_\mathcal{K}\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_\mathrm{M} (\mathfrak{J},\mathcal{K}\times\mathcal{K})$.

Proof: Let $\emptyset_\mathcal{K} = (\mathfrak{Q},\mathcal{K})$, where for all $\mathscr{k}\in\mathcal{K}$, $\mathfrak{Q}(\mathscr{k}) = \emptyset$. Assume that $(\mathfrak{J},\mathcal{K})\Lambda_\gamma\emptyset_\mathcal{K} = (\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{K}) = (\mathfrak{S},\mathcal{K}\times\mathcal{K})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{K}$, $\mathfrak{S}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{Q}(z) = \mathfrak{J}'(\mathscr{k})\cap\emptyset = \emptyset$. Thus, $(\mathfrak{S},\mathcal{K}\times\mathcal{K}) = (\emptyset,\mathcal{K}\times\mathcal{K})$. Let $\emptyset_\mathcal{K}\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_\mathrm{M} (\mathfrak{N},\mathcal{K}\times\mathcal{K})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{K}$,$\mathfrak{N}(\mathscr{k},z) = \mathfrak{Q}'(\mathscr{k})\cap\mathfrak{J}(z) = \mathrm{U}\cap\mathfrak{J}(z) = \mathfrak{J}(z)$, implying that $(\mathfrak{N},\mathcal{K}\times\mathcal{K}) = (\mathfrak{J},\mathcal{K}\times\mathcal{K})$.

Proposition 3.11. *Let* $(\mathfrak{J},\mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma\mathrm{U}_\mathcal{K} =_\mathrm{M} (\mathfrak{J},\mathcal{K})^r$ *and* $\mathrm{U}_\mathcal{K}\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_\mathrm{M} \emptyset_{\mathcal{K}\times\mathcal{K}}$.

Proof: Let $\mathrm{U}_\mathcal{K} = (\mathfrak{Q},\mathcal{K})$, where for all $\mathscr{k}\in\mathcal{K}$, $\mathfrak{Q}(\mathscr{k}) = \mathrm{U}$. Assume that $(\mathfrak{J},\mathcal{K})\Lambda_\gamma\mathrm{U}_\mathcal{K} = (\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{K}) = (\mathfrak{S},\mathcal{K}\times\mathcal{K})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{K}$, $\mathfrak{S}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{Q}(z) = \mathfrak{J}'(\mathscr{k})\cap\mathrm{U} = \mathfrak{J}'(\mathscr{k})$. Thus, $(\mathfrak{S},\mathcal{K}\times\mathcal{K}) = (\mathfrak{J},\mathcal{K}\mathrm{x}\mathcal{K})^r$. Let $\mathrm{U}_\mathcal{K}\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_\mathrm{M} (\mathfrak{P},\mathcal{K}\mathrm{x}\mathcal{K})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{K}$, $\mathfrak{P}(\mathscr{k},z) = \mathfrak{Q}'(\mathscr{k})\cap\mathfrak{J}(z) = \emptyset\cap\mathfrak{J}(z) = \emptyset$ implying that $(\mathfrak{P},\mathcal{K}\mathrm{x}\mathcal{K}) = \emptyset_{\mathcal{K}\times\mathcal{K}}$.

Proposition 3.12. *Let* $(\mathfrak{J},\mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_\mathrm{J} (\mathfrak{J},\mathcal{K})$. *That is,* Λ_γ*-product is not idempotent in* $\mathrm{S_E(U)}$ *under J-equality.*

Proof: Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) = (\mathfrak{S},\mathcal{K}\times\mathcal{K})$, where for all $(\mathscr{k},z)\in\mathcal{K}x\mathcal{K}$, $\mathfrak{S}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{J}(z)$. Since for all $(\mathscr{k},z)\in\mathcal{K}x\mathcal{K}$, there exists $z\in\mathcal{K}$ such that $\mathfrak{S}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{J}(z)\subseteq\mathfrak{J}(z)$, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_\mathrm{J} (\mathfrak{J},\mathcal{K})$ is obtained.

Proposition 3.13. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_\mathrm{J} (\mathfrak{J},\mathcal{K})^\mathrm{r}$ *and* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_\mathrm{J} (\mathfrak{S},\mathcal{Z})$.

Proof: Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{Q}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{S}(z)$. Since for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, there exists $\mathscr{k}\in\mathcal{K}$ such that $\mathfrak{J}'(\mathscr{k})\cap\mathfrak{S}(z)\subseteq\mathfrak{J}'(\mathscr{k})$, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_\mathrm{J} (\mathfrak{J},\mathcal{K})^\mathrm{r}$ is obtained. Similarly, since for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, there exists $z\in\mathcal{Z}$ such that $\mathfrak{J}'(\mathscr{k})\cap\mathfrak{S}(z) \subseteq\mathfrak{S}(z)(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_\mathrm{J} (\mathfrak{S},\mathcal{Z})$ is obtained.

Proposition 3.14. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\right]^\mathrm{r} = (\mathfrak{J},\mathcal{K})^\mathrm{r} \vee_+ (\mathfrak{S},\mathcal{Z})^\mathrm{r}$.

Proof: Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{Q}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{S}(z)$. Hence, $\mathfrak{Q}'(\mathscr{k},z) = (\mathfrak{J}')'(\mathscr{k})\cup(\mathfrak{S})'(z)$, implying that $(\mathfrak{Q}',\mathcal{K}\times\mathcal{Z}) = (\mathfrak{J},\mathcal{K})^\mathrm{r} \vee_+ (\mathfrak{S},\mathcal{Z})^\mathrm{r}$. (For more about $\vee_+$-product, please see [92]).

Proposition 3.15. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_\mathrm{F} (\mathfrak{J},\mathcal{K}) \vee_+ (\mathfrak{S},\mathcal{Z})$.

Proof: Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\zeta,\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K}) \vee_+ (\mathfrak{S},\mathcal{Z}) = (\mathfrak{E},\mathcal{K}\times\mathcal{Z})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\zeta(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{S}(z)$ and for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{E}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cup\mathfrak{S}(z)$. Thus, for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\zeta(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k})\cap\mathfrak{S}(z)\subseteq\mathfrak{J}'(\mathscr{k})\cup\mathfrak{S}(z) = \mathfrak{E}(\mathscr{k},z)$. This completes the proof.

Proposition 3.16. *Let* $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ *and* $(\mathfrak{Q},\mathcal{C})$ *be SSs over* U. *If* $(\mathfrak{J},\mathcal{K})^r \tilde{\subseteq}_\mathrm{F} (\mathfrak{S},\mathcal{Z})^r$, *then* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C}) \tilde{\subseteq}_\mathrm{F} (\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})$.

Proof: Let $(\mathfrak{J},\mathcal{K})^r \tilde{\subseteq}_\mathrm{F} (\mathfrak{S},\mathcal{Z})^r$. Thus, $\mathcal{K}\subseteq\mathcal{Z}$ and for all $\mathscr{k}\in\mathcal{K}$, $\mathfrak{J}'(\mathscr{k})\subseteq\mathfrak{S}'(\mathscr{k})$. Thus, $\mathcal{K}\times\mathcal{C}\subseteq\mathcal{Z}\times\mathcal{C}$ and for all $(\mathscr{k},c)\in\mathcal{K}\times\mathcal{C}$, $\mathfrak{J}'(\mathscr{k})\cap\mathfrak{Q}(c)\subseteq\mathfrak{S}'(\mathscr{k})\cap\mathfrak{Q}(c)$. This completes the proof.

Proposition 3.17. *Let* $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z}),(\mathfrak{Q},\mathcal{C})$ *and* $(\mathfrak{G},\mathcal{W})$ *be SSs over* U. *If* $(\mathfrak{J},\mathcal{K})^r \tilde{\subseteq}_\mathrm{F} (\mathfrak{S},\mathcal{Z})^r$ *and* $(\mathfrak{Q},\mathcal{C}) \tilde{\subseteq}_\mathrm{F} (\mathfrak{G},\mathcal{W})$, *then* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C}) \tilde{\subseteq}_\mathrm{F} (\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{G},\mathcal{W})$.

Proof: Let $(\mathfrak{J},\mathcal{K})^r \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S},\mathcal{Z})^r$ and $(\mathfrak{Q},\mathcal{C}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{d},\mathcal{W})$. Thus, $\mathcal{K} \subseteq \mathcal{Z}, \mathcal{C} \subseteq \mathcal{W}$, for all $k \in \mathcal{K}$, $\mathfrak{J}'(k) \subseteq \mathfrak{S}'(k)$ and for all $c \in \mathcal{C}$, $\mathfrak{Q}(c) \subseteq \mathfrak{d}(c)$. Hence, $\mathcal{K} \times \mathcal{C} \subseteq \mathcal{Z} \times \mathcal{W}$, and for all $(k,c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{J}'(k) \cap \mathfrak{Q}(c) \subseteq \mathfrak{S}'(k) \cap \mathfrak{d}(c)$. This completes the proof.

Proposition 3.18. *Let* $(\mathfrak{J},\mathcal{K}), (\mathfrak{S},\mathcal{K}), (\mathfrak{Q},\mathcal{K})$ *and* $(\mathfrak{d},\mathcal{K})$ *be SSs over* U. *If* $(\mathfrak{S},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{Q},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{d},\mathcal{K})$, *then* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S},\mathcal{K})\Lambda_\gamma(\mathfrak{d},\mathcal{K})$.

Proof: Let $(\mathfrak{S},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{J},\mathcal{K})$ and $(\mathfrak{Q},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{d},\mathcal{K})$, where for all $k \in \mathcal{K}$, $\mathfrak{J}'(k) \subseteq \mathfrak{S}'(k)$ and for all $\ell \in \mathcal{K}\mathfrak{Q}(\ell) \subseteq \mathfrak{d}(\ell)$. Thus, for all $(k,\ell) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{J}'(k) \cap \mathfrak{Q}(\ell) \subseteq \mathfrak{S}'(k) \cap \mathfrak{d}(\ell)$, completing the proof.

Proposition 3.19. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $\emptyset_{\mathcal{K}\times\mathcal{Z}} \tilde{\subseteq}_{\mathrm{F}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$ *and* $\emptyset_{\mathcal{Z}\times\mathcal{K}} \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})$.

Proof: Let $\emptyset_{\mathcal{K}\times\mathcal{Z}} = (\mathcal{E},\mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathcal{S},\mathcal{K} \times \mathcal{Z})$, where for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(k,z) = \emptyset$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{S}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z)$. Since $\mathcal{K} \times \mathcal{Z} \subseteq \mathcal{K} \times \mathcal{Z}$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(k,z) = \emptyset \subseteq \mathfrak{J}'(k) \cap \mathfrak{S}(z) = \mathcal{S}(k,z)$, $\emptyset_{\mathcal{K}\times\mathcal{Z}} \tilde{\subseteq}_{\mathrm{F}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$ is obtained. Similarly, $\emptyset_{\mathcal{Z}\times\mathcal{K}} \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})$.

Proposition 3.20. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $\emptyset_{\mathcal{K}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$, $\emptyset_{\mathcal{Z}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$ *and* $\emptyset_{\mathrm{E}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$.

Proof: Let $\emptyset_{\mathcal{K}} = (\mathcal{E},\mathcal{K})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathcal{S},\mathcal{K} \times \mathcal{Z})$, where for all $k \in \mathcal{K}, \mathcal{E}(k) = \emptyset$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{S}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z)$. Since for all $k \in \mathcal{K}$, there exists $(k,z) \in \mathcal{K} \times \mathcal{Z}$ such that $\mathcal{E}(k) = \emptyset \subseteq \mathfrak{J}'(k) \cap \mathfrak{S}(z) = \mathcal{S}(k,z)$, $\emptyset_{\mathcal{K}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$ is obtained. Similarly, $\emptyset_{\mathcal{Z}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$ and $\emptyset_{\mathrm{E}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$ is obtained.

Proposition 3.21. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{K}\times\mathcal{Z}}$ *and* $(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{Z}\times\mathcal{K}}$.

Proof: Let $\mathrm{U}_{\mathcal{K}\times\mathcal{Z}} = (\mathfrak{Q},\mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathfrak{d},\mathcal{K} \times \mathcal{Z})$, where for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{Q}(k,z) = \mathrm{U}$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{d}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z)$. Since $\mathcal{K} \times \mathcal{Z} \subseteq \mathcal{K} \times \mathcal{Z}$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{d}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z) \subseteq \mathrm{U}{=}\mathfrak{Q}(k,z)$, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{K}\times\mathcal{Z}}$ is obtained. Similarly, $(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{Z}\times\mathcal{K}}$.

Proposition 3.22. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{K}}$ *and* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{Z}}$.

Proof: Let $\mathrm{U}_{\mathcal{K}} = (\mathcal{W},\mathcal{K})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathcal{X},\mathcal{K} \times \mathcal{Z})$, where for all $k \in \mathcal{K}$, $\mathcal{W}(k) = \mathrm{U}$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z)$. Since for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, there exists $k \in \mathcal{K}$ such that $\mathcal{X}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z) \subseteq \mathrm{U} = \mathcal{W}(k)$, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{K}}$ is obtained. Similarly, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{Z}}$.

Proposition 3.23. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}\times\mathcal{Z}}$ *if and only if* $(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} \emptyset_{\mathcal{K}}$ *and* $(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{Z}}$.

Proof: Let $\mathrm{U}_{\mathcal{K}\times\mathcal{Z}} = (\mathcal{E},\mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathcal{X},\mathcal{K} \times \mathcal{Z})$, where for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(k,z) = \mathrm{U}$ and for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(k,z) = \mathfrak{J}'(k) \cap \mathfrak{S}(z)$. Let $(\mathcal{E},\mathcal{K} \times \mathcal{Z}) = (\mathcal{X},\mathcal{K} \times \mathcal{Z})$. Thus, for all $(k,z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{J}'(k) \cap \mathfrak{S}(z) = \mathrm{U}$, implying that for all $k \in \mathcal{K}$, $\mathfrak{J}'(k) = \mathrm{U}$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = \mathrm{U}$. Therefore, for all $k \in \mathcal{K}$, $\mathfrak{J}(k) = \emptyset$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = \mathrm{U}$. Thus, $(\mathfrak{J},\mathcal{K}) = \emptyset_{\mathcal{K}}$ and $(\mathfrak{S},\mathcal{Z}) = \mathrm{U}_{\mathcal{Z}}$.

Conversely, let $(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} \emptyset_{\mathcal{K}}$ and $(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{Z}}$. Thus, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) = \emptyset$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = \mathrm{U}$. Hence, for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(z) = \mathrm{U} \cap \mathrm{U} = \mathrm{U}$, implying that $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}\times\mathcal{Z}}$.

Proposition 3.24. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \emptyset_\emptyset$ *if and only if* $(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} \emptyset_\emptyset$ *or* $(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \emptyset_\emptyset$.

Proof: Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \emptyset_\emptyset$. Thereby, $\mathcal{K} \times \mathcal{Z} = \emptyset$, and so $\mathcal{K} = \emptyset$ or $\mathcal{Z} = \emptyset$. Since $\emptyset_\emptyset$ is the only SS with the empty PS, $(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} \emptyset_\emptyset$ or $(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \emptyset_\emptyset$.

Conversely, let $(\mathfrak{J},\mathcal{K}) = \emptyset_\emptyset$ or $(\mathfrak{S},\mathcal{Z}) = \emptyset_\emptyset$. Then, $\mathcal{K} = \emptyset$ or $\mathcal{Z} = \emptyset$. Since $\mathcal{K} \times \mathcal{Z} = \emptyset$ and $\emptyset_\emptyset$ is the only SS with empty PS, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) =_{\mathrm{M}} \emptyset_\emptyset$.

4. Distributions of Soft Gamma-Product over Certain Types of Soft Set's Operations

In this section, we explore the distributions of soft gamma-product over restricted, extended, soft binary piecewise intersection and union operations, AND-product and OR-product.

Theorem 4.1. *Let* $(\mathfrak{J},\mathcal{K}), (\mathfrak{S},\mathcal{Z})$ *and* $(\mathfrak{Q},\mathcal{C})$ *be SSs over* **U***. Then, we have the following distributions of soft gamma-product over restricted intersection and union operations*:

i) $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})] \cup_{\mathrm{R}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})]$.

ii) $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})] \cap_{\mathrm{R}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})]$.

iii) $[(\mathfrak{S},\mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})]\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})] \cup_{\mathrm{R}} [(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]$.

iv) $[(\mathfrak{S},\mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})]\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})] \cap_{\mathrm{R}} [(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]$.

Proof: **(i)** The PS of the left-hand side (LHS) is $\mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C})$, and the PS of the righthand side (RHS) is $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})$. Since $\mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C}) = (\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})$, the first condition of the M-equality is satisfied. Let $(\mathfrak{S},\mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cap \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}$, $\mathfrak{E}(\mathfrak{z}) = \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z})$. Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{E},\mathcal{Z} \cap \mathcal{C}) = (\wp, \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}))$, where for all $(\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})$, $\wp(\mathscr{k},\mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{E}(\mathfrak{z})$. Thus,

$$\wp(\mathscr{k},\mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cap [\mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z})].$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathfrak{M},\mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K} \times \mathcal{C})$, where for all $(\mathscr{k},z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{M}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(z)$ and for all $(\mathscr{k},c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{P}(\mathscr{k},c) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{Q}(c)$. Let $(\mathfrak{M},\mathcal{K} \times \mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{P},\mathcal{K} \times \mathcal{C}) = (\mathfrak{R},(\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}))$, where for all $(\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \mathfrak{M}(\mathscr{k},\mathfrak{z}) \cup \mathfrak{P}(\mathscr{k},\mathfrak{z}) = [\mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(\mathfrak{z})] \cup [\mathfrak{J}'(\mathscr{k}) \cap \mathfrak{Q}(\mathfrak{z})].$$

Thus, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})] \cup_{\mathrm{R}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})]$.

Here, if $\mathcal{Z} \cap \mathcal{C} = \emptyset$, then $\mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C}) = (\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C}) = \emptyset$. Since the only SS with an empty PS is $\emptyset_\emptyset$, then both sides are $\emptyset_\emptyset$. Since $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C}) = \mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C})$, if $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})=\emptyset$, then $\mathcal{K}=\emptyset$ or $\mathcal{Z} \cap \mathcal{C} = \emptyset$. By assumption, $\mathcal{K} \neq \emptyset$. Thus, $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})=\emptyset$ implies that $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Therefore, under this condition, both sides are again $\emptyset_\emptyset$.

(iii) The PS of the LHS is $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, and the PS of the RHS is $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})$, and since $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})$, the first condition of M-equality is satisfied. Let $(\mathfrak{S},\mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q},\mathcal{C}) =$

$(\mathfrak{E}, \mathcal{Z} \cap \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}$, $\mathfrak{E}(\mathfrak{z}) = \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})$. Let $(\mathfrak{E}, \mathcal{Z} \cap \mathcal{C})\Lambda_\gamma(\mathfrak{J}, \mathcal{K}) = (\wp, (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}))$, where for all $(\mathfrak{z}, \mathscr{k}) \in (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, $\wp(\mathfrak{z}, \mathscr{k}) = \mathfrak{E}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})$. Thus,

$$\wp(\mathfrak{z}, \mathscr{k}) = [\mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})]' \cap \mathfrak{J}(\mathscr{k}) = \left[\mathfrak{S}'^{(\mathfrak{z})} \cup \mathfrak{Q}'^{(\mathfrak{z})}\right] \cap \mathfrak{J}(\mathscr{k}).$$

Let $(\mathfrak{S}, \mathcal{Z})\Lambda_\gamma(\mathfrak{J}, \mathcal{K}) = (\mathfrak{M}, \mathcal{Z} \times \mathcal{K})$ and $(\mathfrak{Q}, \mathcal{C})\Lambda_\gamma(\mathfrak{J}, \mathcal{K}) = (\mathfrak{P}, \mathcal{C} \times \mathcal{K})$, where for all $(z, \mathscr{k}) \in \mathcal{Z} \times \mathcal{K}$, $\mathfrak{M}(z, \mathscr{k}) = \mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})$ and for all $(c, \mathscr{k}) \in \mathcal{C} \times \mathcal{K}$, $\mathfrak{P}(c, \mathscr{k}) = \mathfrak{Q}'(c) \cap \mathfrak{J}(\mathscr{k})$. Assume that $(\mathfrak{M}, \mathcal{Z} \times \mathcal{K}) \cup_R (\mathfrak{P}, \mathcal{C} \times \mathcal{K}) = (\mathfrak{R}, (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}))$, where for all $(\mathfrak{z}, \mathscr{k}) \in (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$,

$$\mathfrak{R}(\mathfrak{z}, \mathscr{k}) = \mathfrak{M}(\mathfrak{z}, \mathscr{k}) \cup \mathfrak{P}(\mathfrak{z}, \mathscr{k}) = [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})] \cup [\mathfrak{Q}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})].$$

Thus, $[(\mathfrak{S}, \mathcal{Z}) \cap_R (\mathfrak{Q}, \mathcal{C})]\Lambda_\gamma(\mathfrak{J}, \mathcal{K}) =_M [(\mathfrak{S}, \mathcal{Z})\Lambda_\gamma(\mathfrak{J}, \mathcal{K})] \cup_R [(\mathfrak{Q}, \mathcal{C})\Lambda_\gamma(\mathfrak{J}, \mathcal{K})]$.

Here if $\mathcal{Z} \cap \mathcal{C} = \emptyset$, then $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$. Since the only SS with the empty PS is $\emptyset_\emptyset$, both sides of the equality are $\emptyset_\emptyset$. Similarly since $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, if $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$, then $\mathcal{Z} \cap \mathcal{C} = \emptyset$ or $\mathcal{K} = \emptyset$. By assumption $\mathcal{K} \neq \emptyset$. Hence, $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$ implies that $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Thus, under this condition, both sides of the equality are again $\emptyset_\emptyset$.

Note 4.2. The restricted SS operation cannot distribute over soft gamma-product as the intersection does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 4.3. *Let* $(\mathfrak{J}, \mathcal{K}), (\mathfrak{S}, \mathcal{Z})$ *and* $(\mathfrak{Q}, \mathcal{C})$ *be SSs over* U. *Then, we have the following distributions of soft gamma-product over extended intersection and union operations*:

i) $(\mathfrak{J}, \mathcal{K})\Lambda_\gamma[(\mathfrak{S}, \mathcal{Z}) \cap_\varepsilon (\mathfrak{Q}, \mathcal{C})] =_M [(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{S}, \mathcal{Z})] \cap_\varepsilon [(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{Q}, \mathcal{C})]$.

ii) $(\mathfrak{J}, \mathcal{K})\Lambda_\gamma[(\mathfrak{S}, \mathcal{Z}) \cup_\varepsilon (\mathfrak{Q}, \mathcal{C})] =_M [(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{S}, \mathcal{Z})] \cup_\varepsilon [(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{Q}, \mathcal{C})]$.

iii) $[(\mathfrak{S}, \mathcal{Z}) \cup_\varepsilon (\mathfrak{Q}, \mathcal{C})]\Lambda_\gamma(\mathfrak{J}, \mathcal{K}) =_M [(\mathfrak{S}, \mathcal{Z})\Lambda_\gamma(\mathfrak{J}, \mathcal{K})] \cap_\varepsilon [(\mathfrak{Q}, \mathcal{C})\Lambda_\gamma(\mathfrak{J}, \mathcal{K})]$.

iv) $[(\mathfrak{S}, \mathcal{Z}) \cap_\varepsilon (\mathfrak{Q}, \mathcal{C})]\Lambda_\gamma(\mathfrak{J}, \mathcal{K}) =_M [(\mathfrak{S}, \mathcal{Z})\Lambda_\gamma(\mathfrak{J}, \mathcal{K})] \cup_\varepsilon [(\mathfrak{Q}, \mathcal{C})\Lambda_\gamma(\mathfrak{J}, \mathcal{K})]$.

Proof: (i) The PS of the LHS is $\mathcal{K}\text{x}(\mathcal{Z} \cup \mathcal{C})$, and the PS of the RHS is $(\mathcal{K}\text{x}\mathcal{Z}) \cup (\mathcal{K}\text{x}\mathcal{C})$. Since $\mathcal{K}\text{x}(\mathcal{Z} \cup \mathcal{C}) = (\mathcal{K}\text{x}\mathcal{Z}) \cup (\mathcal{K}\text{x}\mathcal{C})$, the first condition of the M-equality is satisfied. As $\mathcal{K} \neq \emptyset$, $\mathcal{Z} \neq \emptyset$ and $\mathcal{C} \neq \emptyset$, $\mathcal{K}\text{x}(\mathcal{Z} \cup \mathcal{C}) \neq \emptyset$ and $(\mathcal{K}\text{x}\mathcal{Z}) \cup (\mathcal{K}\text{x}\mathcal{C}) \neq \emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S}, \mathcal{Z}) \cap_\varepsilon (\mathfrak{Q}, \mathcal{C}) = (\mathfrak{E}, \mathcal{Z} \cup \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cup \mathcal{C}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C} \\ \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{C} - \mathcal{Z} \\ \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Let $(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{E}, \mathcal{Z} \cup \mathcal{C}) = (\mathbb{Q}, \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C}))$, where for all $(\mathscr{k}, \mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$, $\mathbb{Q}(\mathscr{k}, \mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{E}(\mathfrak{z})$. Thus, for all $(\mathscr{k}, \mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$,

$$\mathbb{Q}(\mathscr{k}, \mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(\mathfrak{z}), & (\mathscr{k}, \mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} - \mathcal{C}) \\ \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{Q}(\mathfrak{z}), & (\mathscr{k}, \mathfrak{z}) \in \mathcal{K} \times (\mathcal{C} - \mathcal{Z}) \\ \mathfrak{J}'(\mathscr{k}) \cap [\mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})], & (\mathscr{k}, \mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}). \end{cases}$$

Let $(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{S}, \mathcal{Z}) = ((\mathfrak{M}, \mathcal{K} \times \mathcal{Z}))$ and $(\mathfrak{J}, \mathcal{K})\Lambda_\gamma(\mathfrak{Q}, \mathcal{C}) = (\mathfrak{P}, \mathcal{K} \times \mathcal{C})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{M}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(z)$ and for all $(\mathscr{k}, c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{P}(\mathscr{k}, c) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{Q}(c)$. Assume that $(\mathfrak{M}, \mathcal{K} \times \mathcal{Z}) \cap_\varepsilon (\mathfrak{P}, \mathcal{K} \times \mathcal{C}) = (\mathfrak{R}, (\mathcal{K} \times \mathcal{Z}) \cup (\mathcal{K} \times \mathcal{C}))$, where for all $(\mathscr{k}, \mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cup (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z})=\begin{cases}\mathfrak{M}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}-\mathcal{C})\\ \mathfrak{P}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{C})-(\mathcal{K}\times\mathcal{Z})=\mathcal{K}\times(\mathcal{C}-\mathcal{Z})\\ \mathfrak{M}(\mathcal{k},\mathfrak{z})\cap\mathfrak{P}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}).\end{cases}$$

Thus,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z})=\begin{cases}\mathfrak{J}'(\mathcal{k})\cap\mathfrak{S}(\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}-\mathcal{C})\\ \mathfrak{J}'(\mathcal{k})\cap\mathfrak{Q}(\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{C})-(\mathcal{K}\times\mathcal{Z})=\mathcal{K}\times(\mathcal{C}-\mathcal{Z})\\ [\mathfrak{J}'(\mathcal{k})\cap\mathfrak{S}(\mathfrak{z})]\cap[\mathfrak{J}'(\mathcal{k})\cap\mathfrak{Q}(\mathfrak{z})], & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}).\end{cases}$$

Hence, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\cap_\varepsilon(\mathfrak{Q},\mathcal{C})]=_M\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\right]\cap_\varepsilon\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})\right]$.

(iii) The PS of the LHS is $(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}$, and the PS of the RHS is $(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})$, and since $(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}=(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})$ the first condition of M-equality is satisfied. By assumption, $\mathcal{K}\neq\emptyset$, $\mathcal{Z}\neq\emptyset$, and $\mathcal{C}\neq\emptyset$. Thus, $(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}\neq\emptyset$ and $(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})\neq\emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z})\cup_\varepsilon(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\cup\mathcal{C})$, where for all $\mathfrak{z}\in\mathcal{Z}\cup\mathcal{C}$,

$$\mathfrak{E}(\mathfrak{z})=\begin{cases}\mathfrak{S}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}-\mathcal{C}\\ \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{C}-\mathcal{Z}\\ \mathfrak{S}(\mathfrak{z})\cup\mathfrak{Q}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}\cap\mathcal{C}.\end{cases}$$

Let $(\mathfrak{E},\mathcal{Z}\cup\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})=(\wp,(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K})$, where for all $(\mathfrak{z},\mathcal{k})\in(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}$, $\wp(\mathfrak{z},\mathcal{k})=\mathfrak{E}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k})$,

$$\wp(\mathcal{k},\mathfrak{z})=\begin{cases}\mathfrak{S}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ \mathfrak{Q}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{C}-\mathcal{Z})\times\mathcal{K}\\ [\mathfrak{S}'(\mathfrak{z})\cap\mathfrak{Q}'(\mathfrak{z})]\cap\mathfrak{J}(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}.\end{cases}$$

Let $(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})=(\mathfrak{M},\mathcal{Z}\times\mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})=(\mathfrak{P},\mathcal{C}\times\mathcal{K})$, where for all $(z,\mathcal{k})\in\mathcal{Z}\times\mathcal{K}$, $\mathfrak{M}(z,\mathcal{k})=\mathfrak{S}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k})$ and for all $(c,\mathcal{k})\in\mathcal{C}\times\mathcal{K}$, $\mathfrak{P}(c,\mathcal{k})=\mathfrak{Q}'(c)\cap\mathfrak{J}(\mathcal{k})$. Assume that $(\mathfrak{M},\mathcal{Z}\times\mathcal{K})\cap_\varepsilon(\mathfrak{P},\mathcal{C}\times\mathcal{K})=\left(\mathfrak{R},(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})\right)$, where for all $(\mathfrak{z},\mathcal{k})\in(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}$,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z})=\begin{cases}\mathfrak{M}(\mathfrak{z},\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ \mathfrak{P}(\mathfrak{z},\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{C}\times\mathcal{K})-(\mathcal{Z}\times\mathcal{K})=(\mathcal{C}-\mathcal{Z})\times\mathcal{K}\\ \mathfrak{M}(\mathfrak{z},\mathcal{k})\cap\mathfrak{P}(\mathfrak{z},\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}.\end{cases}$$

Thus,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z})=\begin{cases}\mathfrak{S}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ \mathfrak{Q}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{C}\times\mathcal{K})-(\mathcal{Z}\times\mathcal{K})=(\mathcal{C}-\mathcal{Z})\times\mathcal{K}\\ [\mathfrak{S}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k})]\cap[\mathfrak{Q}'(\mathfrak{z})\cap\mathfrak{J}(\mathcal{k})], & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}.\end{cases}$$

Hence, $[(\mathfrak{S},\mathcal{Z})\cup_\varepsilon(\mathfrak{Q},\mathcal{C})]\Lambda_\gamma(\mathfrak{J},\mathcal{K})=_M\left[(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})\right]\cap_\varepsilon\left[(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})\right]$.

Note 4.4. The extended SS operation cannot distribute over soft gamma-product as the union operation does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 4.5. *Let* $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})$ *and* $(\mathfrak{Q},\mathcal{C})$ *be SSs over* U. *Then, we have the following distributions of soft gamma-product over soft binary piecewise intersection and union operations*:

i) $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\,\tilde{\cap}\,(\mathfrak{Q},\mathcal{C})]=_M\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\right]\tilde{\cap}\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})\right]$.

ii) $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{Q},\mathcal{C})]=_M\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})\right]\tilde{\cup}\left[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})\right]$.

iii) $[(\mathfrak{S},\mathcal{Z})\ \tilde{\cup}\ (\mathfrak{Q},\mathcal{C})]\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]\ \tilde{\cap}\ [(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]$.

iv) $[(\mathfrak{S},\mathcal{Z})\ \tilde{\cap}\ (\mathfrak{Q},\mathcal{C})]\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]\ \tilde{\cup}\ [(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]$.

Proof: (i) Since the PS of the SSs of both sides are $\mathcal{K}\mathrm{x}\mathcal{Z}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{K} \neq \emptyset$ and $\mathcal{Z} \neq \emptyset$ by assumption, $\mathcal{K}\mathrm{x}\mathcal{Z} \neq \emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z})\ \tilde{\cap}\ (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z})$, where for all $\mathfrak{z} \in \mathcal{Z}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C} \\ \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{E},\mathcal{Z}) = (\mathbb{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(\mathscr{k},\mathfrak{z}) \in \mathcal{K}\times\mathcal{Z}$, $\mathbb{Q}(\mathscr{k},\mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{E}(\mathfrak{z})$. Thus,

$$\mathbb{Q}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in \mathcal{K}\times(\mathcal{Z}-\mathcal{C}) \\ \mathfrak{J}'(\mathscr{k}) \cap [\mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})], & (\mathscr{k},\mathfrak{z}) \in \mathcal{K}\times\mathcal{Z}\cap\mathcal{C}. \end{cases}$$

Assume that $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) = (\mathfrak{M},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K}\times\mathcal{C})$, where for all $(\mathscr{k},z) \in \mathcal{K}\times\mathcal{Z}$, $\mathfrak{M}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(\mathfrak{z})$ and for all $(\mathscr{k},c) \in \mathcal{K}\times\mathcal{C}$, $\mathfrak{P}(\mathscr{k},c) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{Q}(c)$. Suppose that $(\mathfrak{M},\mathcal{K}\times\mathcal{Z})\ \tilde{\cap}\ (\mathfrak{P},\mathcal{K}\times\mathcal{C}) = (\mathfrak{R},(\mathcal{K}\times\mathcal{Z}))$, where for all$(\mathscr{k},\mathfrak{z}) \in \mathcal{K}\times\mathcal{Z}$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathscr{k},\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}-\mathcal{C}) \\ \mathfrak{M}(\mathscr{k},\mathfrak{z}) \cap \mathfrak{P}(\mathscr{k},\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}). \end{cases}$$

Thus,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}-\mathcal{C}) \\ [\mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(\mathfrak{z})] \cap [\mathfrak{J}'(\mathscr{k}) \cap \mathfrak{Q}(\mathfrak{z})], & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}). \end{cases}$$

Hence, $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\ \tilde{\cap}\ (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})]\ \tilde{\cap}\ [(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})]$.

Since $\mathcal{K} \neq \mathcal{K}\mathrm{x}\mathcal{K}$, the soft binary piecewise operations do not distribute over soft gamma-product operations.

(iii) Since the PS of the SSs of both sides are $\mathcal{Z}\times\mathcal{K}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{Z} \neq \emptyset$ and $\mathcal{K} \neq \emptyset$ by assumption, $\mathcal{Z}\times\mathcal{K} \neq \emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z})\ \tilde{\cup}\ (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z})$, where for all $\mathfrak{z} \in \mathcal{Z}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C} \\ \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Let $(\mathfrak{E},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) = (\wp,\mathcal{Z}\times\mathcal{K})$, where for all $(\mathfrak{z},\mathscr{k}) \in \mathcal{Z}\times\mathcal{K}$, $\wp(\mathfrak{z},\mathscr{k}) = \mathfrak{E}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})$.Thus,

$$\wp(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z}-\mathcal{C})\times\mathcal{K} \\ [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{Q}'(\mathfrak{z})] \cap \mathfrak{J}(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}. \end{cases}$$

Assume that $(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) = (\mathfrak{M},\mathcal{Z}\times\mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K}) = (\mathfrak{P},\mathcal{C}\times\mathcal{K})$, where for all $(z,\mathscr{k}) \in \mathcal{Z}\times\mathcal{K}$, $\mathfrak{M}(z,\mathscr{k}) = \mathfrak{S}'(z) \cap \mathfrak{J}(\mathscr{k})$ and for all $(c,\mathscr{k}) \in \mathcal{C}\times\mathcal{K}$, $\mathfrak{P}(c,\mathscr{k}) = \mathfrak{Q}'(c) \cap \mathfrak{J}(\mathscr{k})$. Let $(\mathfrak{M},\mathcal{Z}\times\mathcal{K})\ \tilde{\cap}\ (\mathfrak{P},\mathcal{C}\times\mathcal{K}) = (\mathfrak{R},(\mathcal{Z}\times\mathcal{K}))$, where for all $(\mathfrak{z},\mathscr{k}) \in (\mathcal{Z}\times\mathcal{K})$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K}) = (\mathcal{Z}-\mathcal{C})\times\mathcal{K} \\ \mathfrak{M}(\mathfrak{z},\mathscr{k}) \cap \mathfrak{P}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K}) = (\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}. \end{cases}$$

Thus,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K}) = (\mathcal{Z}-\mathcal{C})\times\mathcal{K} \\ [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})] \cap [\mathfrak{Q}'(\mathfrak{z}) \cap \mathfrak{J}(\mathscr{k})], & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K}) = (\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}. \end{cases}$$

Hence, $[(\mathfrak{S},\mathcal{Z})\ \tilde{\cup}\ (\mathfrak{Q},\mathcal{C})]\Lambda_\gamma(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]\ \tilde{\cap}\ [(\mathfrak{Q},\mathcal{C})\Lambda_\gamma(\mathfrak{J},\mathcal{K})]$.

Proposition 4.6. *Let* $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})$ *and* $(\mathfrak{Q},\mathcal{C})$ *be SSs over* U. *Then,*

(1) $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\Lambda(\mathfrak{Q},\mathcal{C})]\,\tilde{\subseteq}_L\,[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})]\Lambda[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})]$.

(2) $(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\vee(\mathfrak{Q},\mathcal{C})]\,\tilde{\subseteq}_L\,[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})]\vee[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})]$.

Proof: (1) Let $(\mathfrak{S},\mathcal{Z})\Lambda(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})$, where for all $(z,c)\in\mathcal{Z}\times\mathcal{C}$, $\mathfrak{E}(z,c)=\mathfrak{S}(z)\cap\mathfrak{Q}(c)$. Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=(\mathfrak{R},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$, where for all $(k,(z,c))\in\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})$,

$$\mathfrak{R}(k,(z,c))=\mathfrak{J}'(k)\cap[\mathfrak{S}(z)\cap\mathfrak{Q}(c)].$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})=(\mathcal{H},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})=(\mathcal{M},\mathcal{K}\times\mathcal{C})$, where for all $(k,z)\in\mathcal{K}\times\mathcal{Z}\mathcal{H}(k,z)=\mathfrak{J}'(k)\cap\mathfrak{S}(z)$ and for all $(k,c)\in\mathcal{K}\times\mathcal{C}$, $\mathcal{M}(k,c)=\mathfrak{J}'(k)\cap\mathfrak{Q}(c)$. Assume that $(\mathcal{H},\mathcal{K}\times\mathcal{Z})\Lambda(\mathcal{M},\mathcal{K}\times\mathcal{C})=(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))$, where for all $((k,z),(k,c))\in(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})$,

$$\beta((k,z),(k,c))=[\mathfrak{J}'(k)\cap\mathfrak{S}(z)]\cap[\mathfrak{J}'(k)\cap\mathfrak{Q}(c)].$$

Here, for all $(k,(z,c))\in\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})$, there exists $((k,z),(k,c))\in(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})$ such that $\mathfrak{R}(k,(z,c))=\mathfrak{J}'(k)\cap[\mathfrak{S}(z)\cap\mathfrak{Q}(c)]=[\mathfrak{J}'(k)\cap\mathfrak{S}(z)]\cap[\mathfrak{J}'(k)\cap\mathfrak{Q}(c)]=\beta((k,z),(k,c))$. This completes the proof. It is obvious that the L-subset in Proposition 4.6 cannot be L-equality with the following example:

Example 4.7. Let $E=\{\ell_1,\ell_2,\ell_3,\ell_4\}$ be the PS, $\mathcal{K}=\{\ell_2,\ell_3\}$, $\mathcal{Z}=\{\ell_1\}$ and $\mathcal{C}=\{\ell_4\}$, be the subsets of E, $U=\{\mathfrak{k}_1,\mathfrak{k}_2,\mathfrak{k}_3,\mathfrak{k}_4,\mathfrak{k}_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ SSs over U such that $(\mathfrak{J},\mathcal{K})=\{(\ell_2,\{\mathfrak{k}_3,\mathfrak{k}_4\}),(\ell_3,\{\mathfrak{k}_2,\mathfrak{k}_3\})\}$, $(\mathfrak{S},\mathcal{Z})=\{(\ell_1,U)\}$ and $(\mathfrak{Q},\mathcal{C})=\{(\ell_4,\{\mathfrak{k}_2,\mathfrak{k}_3,\mathfrak{k}_4\})\}$. We show that

$$(\mathfrak{J},\mathcal{K})\Lambda_\gamma[(\mathfrak{S},\mathcal{Z})\Lambda(\mathfrak{Q},\mathcal{C})]\neq_L[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})]\Lambda[(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})].$$

Let $(\mathfrak{S},\mathcal{Z})\Lambda(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})$, where

$$(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=\{((\ell_1,\ell_4),\{\mathfrak{k}_2,\mathfrak{k}_3,\mathfrak{k}_4\})\}.$$

Assume that $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$, where

$$(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))=\left(\left((\ell_2,(\ell_1,\ell_4)),\{\mathfrak{k}_2\}\right),\left((\ell_3,(\ell_1,\ell_4)),\{\mathfrak{k}_4\}\right)\right).$$

Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})=(\mathfrak{E},\mathcal{K}\times\mathcal{Z})$, where

$$(\mathfrak{E},\mathcal{K}\times\mathcal{Z})=\{((\ell_2,\ell_1),\{\mathfrak{k}_1,\mathfrak{k}_2,\mathfrak{k}_5\}),((\ell_3,\ell_1),\{\mathfrak{k}_1,\mathfrak{k}_4,\mathfrak{k}_5\})\}.$$

Suppose that $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{Q},\mathcal{C})=(\mathfrak{R},\mathcal{K}\times\mathcal{C})$, where

$$(\mathfrak{R},\mathcal{K}\times\mathcal{C})=[((\ell_2,\ell_4),\{\mathfrak{k}_2\}),((\ell_3,\ell_4),\{\mathfrak{k}_4\})].$$

Let $(\mathfrak{E},\mathcal{K}\times\mathcal{Z})\Lambda(\mathfrak{R},\mathcal{K}\times\mathcal{C})=(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))$. Then,

$$(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))=\begin{Bmatrix}\left(((\ell_2,\ell_1),(\ell_2,\ell_4)),\{\mathfrak{k}_2\}\right),\left(((\ell_2,\ell_1),(\ell_3,\ell_4)),\emptyset\right),\\ \left(((\ell_3,\ell_1),(\ell_2,\ell_4)),\emptyset\right),\left(((\ell_3,\ell_1),(\ell_3,\ell_4)),\{\mathfrak{k}_4\}\right).\end{Bmatrix}$$

Thus, $\beta((\ell_2,\ell_1),(\ell_3,\ell_4))\neq\mathfrak{M}(\ell_2,(\ell_1,\ell_4))$, $\beta((\ell_2,\ell_1),(\ell_3,\ell_4))\neq\mathfrak{M}(\ell_3,(\ell_1,\ell_4))$, $\beta((\ell_3,\ell_1),(\ell_2,\ell_4))\neq\mathfrak{M}(\ell_2,(\ell_1,\ell_4))$, $\beta((\ell_3,\ell_1),(\ell_2,\ell_4))\neq\mathfrak{M}(\ell_3,(\ell_1,\ell_4))$, implying that $(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))\,\tilde{\nsubseteq}_L\,(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$. Hence, $(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))\neq_L(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$.

5. *uni-int* Decision-Making Method Applied to Soft Gamma-Product

The *uni-int* decision-making approach is applied in this section by applying the *uni-int* operator and int-uni decision function developed by Çağman and Enginoğlu [11] to the soft gamma-product.

Throughout this section, all the soft gamma-products (Λ_γ) of the SSs overU are assumed to be contained in the set $\Lambda_\gamma(\mathrm{U})$, and the approximation function of the soft gamma-product of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, that is $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z})$

$$\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z}: \mathcal{K}\mathrm{x}\mathcal{Z} \to \mathrm{P(U)},$$

where $(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z})(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(z)$ for all $(\mathscr{k},z) \in \mathcal{K}\mathrm{x}\mathcal{Z}$.

Definition 5.1. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SS over U. Then, *uni-int* operators for soft gamma-product, denoted by uni_xint_y and uni_yint_x are defined respectively as

uni_xint_y: $\Lambda_\gamma \to \mathrm{P(U)}$, $\quad uni_xint_y(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z}) = \cup_{\mathscr{k}\in\mathcal{K}}(\cap_{z\in Z}(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z})(\mathscr{k},z)))$,

$unit_yint_x$: $\Lambda_\gamma \to \mathrm{P(U)}$, $\quad uni_yint_x(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z}) = \cup_{z\in Z}(\cap_{\mathscr{k}\in\mathcal{K}}(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z})(\mathscr{k},z)))$.

Definition 5.2. [11] Let $(\mathfrak{J},\mathcal{K})\Lambda_\gamma(\mathfrak{S},\mathcal{Z}) \in \mathrm{V}_\lambda(\mathrm{U})$. Then, *uni-int* decision function for soft gamma-product, denoted by *uni-int* are defined by

uni-int: $\Lambda_\gamma \to \mathrm{P(U)}$, $\quad$ *uni-int*$(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z}) = uni_xint_y(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z}) \cup uni_yint_x(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z})$.

The values *uni-int*$(\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z})$ is a subset of U called *uni-int* decision set of $\mathfrak{J}_\mathcal{K}\Lambda_\gamma\mathfrak{S}_\mathcal{Z}$.

The *int-uni* decision-making approach may be used in the following ways to choose the best set of alternatives while staying focused on the current issue given a set of parameters and options:

Step 1: From the parameter collection, choose feasible subsets.

Step 2: Create the SSs for every parameter sets.

Step 3: Determine the SSs' soft gamma-product.

Step 4: Create the result of *uni-int* decision set.

This method demonstrates the value of SS theory in handling decision-making scenarios by enabling its application to the *uni-int* decision-making problem, particularly in the setting of soft gamma-product.

Example 5.3. A tour company has issued a recruitment announcement for the position of tour guide, with candidates to be selected based on their interview performance. Applications will first undergo an initial screening to verify compliance with the required qualifications for the position. Any applications that fail to meet the specified criteria will be disqualified. Among the candidates with valid applications, the selection process will be conducted by Mr. Hüseyin, the Human Resources Manager of the company. He will evaluate the interview results by first identifying undesirable characteristics and then focusing on the attributes he values in tour guide candidates. The selected candidates will then undergo a comprehensive training program, and those who successfully complete it will qualify to join the tour company's professional tour guide team. The final decision will be made using the *uni-int* decision-making method applied to the soft gamma-product. Assume the following profiles for the candidates whose applications are deemed valid $\mathrm{U} = \{r_1, r_2, \dots, r_{24}\}$. Let the set of parameters used to determine the selected tour guides be represented as $\{a_1, a_2, \dots, a_8\}$. Each parameter a_i where $i \in \{1,2,\dots 8\}$ corresponds to the following descriptions, respectively:

- a_1 = “having the updated professional knowledge of history, archaeology, art history, geography, and economics required by the profession”,
- a_2 = “having a low level of general understanding of topics like music, art, and politics”,
- a_3 = “having insufficient practical experience in the professional field required by the position”,
- a_4 = “having effective communication”,
- a_5 = “using time efficiently”,
- a_6 = “solution-oriented in addressing problems such as reservation problems, illness, death, theft, accidents”,
- a_7 = “low physical fitness for long walks, extended road trips, sleeplessness, challenging conditions, high altitudes, and varying climate conditions”,
- a_8 = “Unable to foster a spirit of exploration, fun, and friendship”.

To address the tour guide recruitment process, we can utilize the *uni-int* method within the framework of the soft gamma-product as follows:

Step 1: Determining the sets of parameters

The decision-maker, Mr. Hüseyin, begins the selection process by categorizing the parameters from the existing set based on their importance. First, he identifies the parameters he does NOT want to see in the selected candidates (Set 1) and then selects the parameters he considers essential for the candidates (Set 2). This categorization ensures a structured and transparent decision-making process:

1. Parameters that are preferred NOT to be present in the selected candidates:
   These represent undesirable traits or deficiencies that render a candidate unsuitable for selection.
2. Parameters that must absolutely be present in the selected candidates:
   These represent essential traits or skills required for a tour guide. The absence of these parameters disqualifies a candidate.

By organizing the parameters into these two sets, the decision-making process aligns with Mr. Hüseyin’s priorities and expectations. Let these parameter sets be defined as follows:

- $\mathcal{K} = \{a_2, a_3, a_7, a_8\}$: Parameters representing undesirable traits.
- $\mathcal{Z} = \{a_1, a_4, a_5, a_6\}$: Parameters representing essential traits.

Step 2: Constructing the SSs by using the parameter sets determined in Step 1.

Using these parameter sets, the decision-maker constructs the SSs $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{J}, \mathcal{Z})$, respectively:

$$(\mathfrak{J}, \mathcal{K}) = \{(a_2, \{r_2, r_4, r_5, r_8, r_{10}, r_{13}, r_{14}, r_{16}, r_{18}, r_{19}, r_{22}, r_{24}\}), (a_3, \{r_1, r_2, r_3, r_7, r_{10}, r_{16}, r_{20}, r_{22}, r_{24}\}),$$
$$(a_7, \{r_5, r_8, r_{12}, r_{15}, r_{17}, r_{18}, r_{20}, r_{21}, r_{23}\}), (a_8, \{r_2, r_4, r_6, r_{11}, r_{12}, r_{14}, r_{18}, r_{22}, r_{23}, r_{24}\})\}$$

and

$$(\mathfrak{J}, \mathcal{Z}) = \{(a_1, \{r_8, r_{10}, r_{13}, r_{15}, r_{17}, r_{18}, r_{23}, r_{24}\}), (a_4, \{r_2, r_4, r_6, r_8, r_9, r_{12}, r_{16}, r_{18}, r_{20}, r_{21}, r_{22}\}),$$
$$(a_5, \{r_2, r_3, r_5, r_8, r_9, r_{12}, r_{17}, r_{18}\}), (a_6, \{r_8, r_9, r_{10}, r_{13}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}\})\}$$

While $(\mathfrak{J}, \mathcal{K})$ is an SS representing candidates to be eliminated due to undesirable parameters in $\mathcal{K}$, $(\mathfrak{J}, \mathcal{Z})$ is an SS representing candidates close to the ideal by possessing the desired parameters in $\mathcal{Z}$.

Step 3: Determine the Λ_γ-product of SSs:

$$\begin{aligned}\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z = \{&((a_2,a_1),\{r_{15},r_{17},r_{23}\}),((a_2,a_4),\{r_6,r_9,r_{12},r_{20},r_{21}\}),\\
&((a_2,a_5),\{r_3,r_9,r_{12},r_{17}\}),((a_2,a_6),\{r_9,r_{20},r_{21}\}),\\
&((a_3,a_1),\{r_8,r_{13},r_{15},r_{17},r_{18},r_{23}\}),((a_3,a_4),\{r_4,r_6,r_8,r_9,r_{12},r_{18},r_{21}\}),\\
&((a_3,a_5),\{r_5,r_8,r_9,r_{12},r_{17},r_{18}\}),((a_3,a_6),\{r_8,r_9,r_{13},r_{18},r_{19},r_{21}\}),\\
&((a_7,a_1),\{r_{10},r_{13},r_{24}\}),((a_7,a_4),\{r_2,r_4,r_6,r_9,r_{16},r_{22}\}),\\
&((a_7,a_5),\{r_2,r_3,r_9\}),((a_7,a_6),\{r_9,r_{10},r_{13},r_{19},r_{22},r_{24}\}),\\
&((a_8,a_1),\{r_8,r_{10},r_{13},r_{15},r_{17}\}),((a_8,a_4),\{r_8,r_9,r_{16},r_{20},r_{21}\}),\\
&((a_8,a_5),\{r_3,r_5,r_8,r_9,r_{17}\}),((a_8,a_6),\{r_8,r_9,r_{10},r_{13},r_{19},r_{20},r_{21}\})\}\end{aligned}$$

Step 4: Determine the set of *uni-int*($\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z$):

$$uni_\Bbbk - int_z(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z) = \cup_{\Bbbk\in\mathcal{K}}(\cap_{z\in Z}(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(\Bbbk,z))).$$

We determine first $\cap_{z\in Z}\left(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(\Bbbk,z)\right)$:

$$\begin{gathered}(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_2,a_1)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_2,a_4)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_2,a_5)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_2,a_6)\\
=\{r_{15},r_{17},r_{23}\}\cap\{r_6,r_9,r_{12},r_{20},r_{21}\}\cap\{r_3,r_9,r_{12},r_{17}\}\\
\cap\{r_9,r_{20},r_{21}\}=\emptyset.\end{gathered}$$

$$\begin{gathered}(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_3,a_1)\cap\mathfrak{I}(_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_3,a_4)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_3,a_5)\cap\mathfrak{I}(_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_3,a_6)\\
=\{r_8,r_{13},r_{15},r_{17},r_{18},r_{23}\}\cap\{r_4,r_6,r_8,r_9,r_{12},r_{18},r_{21}\}\cap\{r_5,r_8,r_9,r_{12},r_{17},r_{18}\}\\
\cap\{r_8,r_9,r_{13},r_{18},r_{19},r_{21}\}=\{r_8,r_{18}\}.\end{gathered}$$

$$\begin{gathered}(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_7,a_1)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_7,a_4)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_7,a_5)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_7,a_6)\\
=\{r_{10},r_{13},r_{24}\}\cap\{r_2,r_4,r_6,r_9,r_{16},r_{22}\}\cap\{r_2,r_3,r_9\}\\
\cap\{r_9,r_{10},r_{13},r_{19},r_{22},r_{24}\}=\emptyset.\end{gathered}$$

$$\begin{gathered}(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_8,a_1)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_8,a_4)\cap\mathfrak{I}(_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_8,a_5)\cap(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(a_8,a_6)\\
=\{r_8,r_{10},r_{13},r_{15},r_{17}\}\cap\{r_8,r_9,r_{16},r_{20},r_{21}\}\cap\{r_3,r_5,r_8,r_9,r_{17}\}\\
\cap\{r_8,r_9,r_{10},r_{13},r_{19},r_{20},r_{21}\}=\{r_8\}.\end{gathered}$$

Thus,

$$uni_\Bbbk - int_z(\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)=\cup_{\Bbbk\in\mathcal{K}}\left(\cap_{z\in Z}\left((\mathfrak{I}_\mathcal{K}\Lambda_\gamma\mathfrak{I}_Z)(\Bbbk,z)\right)\right)=\emptyset\cup\{r_8,r_{18}\}\cup\emptyset\cup\{r_8\}=\{r_8,r_{18}\}$$

$uni_z - int_{\mathscr{k}}(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z}) = \cup_{z\in Z}(\cap_{\mathscr{k}\in\mathcal{K}}\left((\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(\mathscr{k},z)\right))$.

We determine first $\cap_{\mathscr{k}\in\mathcal{K}}\left((\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(\mathscr{k},z)\right)$:

$$(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_2,a_1)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_3,a_1)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_7,a_1)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_8,a_1)$$
$$=\{r_{15},r_{17},r_{23}\}\cap\{r_8,r_{13},r_{15},r_{17},r_{18},r_{23}\}\cap\{r_{10},r_{13},r_{24}\}$$
$$\cap\{r_8,r_{10},r_{13},r_{15},r_{17}\}=\emptyset.$$

$$(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_2,a_4)\cap\mathfrak{I}({}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_3,a_4)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_7,a_4)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_8,a_4)$$
$$=\{r_6,r_9,r_{12},r_{20},r_{21}\}\cap\{r_4,r_6,r_8,r_9,r_{12},r_{18},r_{21}\}\cap\{r_2,r_4,r_6,r_9,r_{16},r_{22}\}$$
$$\cap\{r_8,r_9,r_{16},r_{20},r_{21}\}=\{r_9\}.$$

$$(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_2,a_5)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_3,a_5)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_7,a_5)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_8,a_5)$$
$$=\{r_3,r_9,r_{12},r_{17}\}\cap\{r_5,r_8,r_9,r_{12},r_{17},r_{18}\}\cap\{r_2,r_3,r_9\}$$
$$\cap\{r_3,r_5,r_8,r_9,r_{17}\}=\{r_9\}.$$

$$(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_2,a_6)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_3,a_6)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_7,a_6)\cap(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(a_8,a_6)$$
$$=\{r_9,r_{20},r_{21}\}\cap\{r_8,r_9,r_{13},r_{18},r_{19},r_{21}\}\cap\{r_9,r_{10},r_{13},r_{19},r_{22},r_{24}\}$$
$$\cap\{r_8,r_9,r_{10},r_{13},r_{19},r_{20},r_{21}\}=\{r_9\}.$$

Thus,

$$uni_z - int_{\mathscr{k}}(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z}) = \cup_{z\in Z}\left(\cap_{\mathscr{k}\in\mathcal{K}}\left((\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})(\mathscr{k},z)\right)\right)=\emptyset\cup\{r_9\}\cup\{r_9\}\cup\{r_9\}=\{r_9\}.$$

Hence,

$$uni\text{-}int(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z}) = [uni_{\mathscr{k}} - int_z(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})]\cup[uni_z - int_{\mathscr{k}}(\mathfrak{I}_{\mathcal{K}}\Lambda_\gamma\mathfrak{I}_{Z})]=\{r_8,r_{18}\}\cup\{r_9\}=\{r_8,r_9,r_{18}\}.$$

From the analysis, it can be concluded that among the 24 applicants whose applications were accepted for the tour company recruitment process, the candidates $\{r_8,r_9,r_{18}\}$ have successfully earned the right to undergo a comprehensive training program of this tour company to join the tour company's professional tour guide team.

6. Conclusion

This study introduced the "soft gamma-product," a new type of soft product developed from Molodtsov's soft set theory. We provided an example with regard to several types of soft subsets and equalities, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality, and thoroughly examined its algebraic properties. We also examined the distributional properties of the soft gamma-product across a number of soft set techniques. We ultimately employed the soft decision-making technique, which simplifies the process by eliminating the need for rough or fuzzy soft sets, to select the optimal components from the range of options to soft gamma-product. Its effectiveness across a range of fields is demonstrated by an example. This work enables a wide range of applications, such as novel algorithms for soft set-based encryption and new methods for decision-making. To improve the soft set literature both conceptually and practically, future research might propose more soft product operations and look into fundamental traits connected to various soft equal connections.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Molodtsov, D. A. (1999). Soft set theory-first results. *Computers and Mathematics with Applications, 37*(4-5), 19-31. https://doi.org/10.1016/S0898-1221(99)00056-5

[3] Maji, P. K., Roy, A. R., & Biswas, R. (2002). An application of soft sets in a decision making problem. *Computers and Mathematics with Applications, 44*(8-9), 1077-1083. https://doi.org/10.1016/S0898-1221(02)00216-X

[4] Chen, D.-G., Tsang, E. C. C., & Yeung, D. S. (2003). Some notes on the parameterization reduction of soft sets. In *Proceedings of the 2003 International Conference on Machine Learning and Cybernetics* (pp. 1442-1445). Xi'an. https://doi.org/10.1109/ICMLC.2003.1259720

[5] Chen, D., Tsang, E. C. C., Yeung, D. S., & Wang, X. (2005). The parametrization reduction of soft sets and its applications. *Computers and Mathematics with Applications, 49*(5-6), 757-763. https://doi.org/10.1016/j.camwa.2004.10.036

[6] Xiao, Z., Chen, L., Zhong, B., & Ye, S. (2005). Recognition for soft information based on the theory of soft sets. In J. Chen (Ed.), *IEEE Proceedings of International Conference on Services Systems and Services Management* (pp. 1104-1106). https://doi.org/10.1109/ICSSSM.2005.1500166

[7] Mushrif, M. M., Sengupta, S., & Ray, A. K. (2006). *Texture classification using a novel, soft-set theory based classification algorithm.* In P. J. Narayanan, S. K. Nayar, & H. T. Shum (Eds.), Computer Vision - ACCV 2006. Lecture Notes in Computer Science (Vol. 3851). Springer, Berlin, Heidelberg. https://doi.org/10.1007/11612032_26

[8] Herawan, M. T., & Deris, M. M. (2009). *A direct proof of every rough set is a soft set*. In Third Asia International Conference on Modelling & Simulation (pp. 119-124). Bundang, Indonesia. https://doi.org/10.1109/AMS.2009.148

[9] Herawan, M. T., & Deris, M. M. (2010). *Soft decision making for patients suspected influenza*. In D. Taniar, O. Gervasi, B. Murgante, E. Pardede, & B. O. Apduhan (Eds.), Computational Science and Its Applications - ICCSA 2010. Lecture Notes in Computer Science (Vol. 6018). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-12179-1_34

[10] Herawan, T. (2005). Soft set-based decision making for patients suspected influenza-like illness. *International Journal of Modern Physics: Conference Series, 1*(1), 1-5.

[11] Çağman, N., & Enginoğlu, S. (2010). Soft set theory and uni-int decision making. *European Journal of Operational Research, 207*(2), 848-855. https://doi.org/10.1016/j.ejor.2010.05.004

[12] Çağman, N., & Enginoğlu, S. (2010). Soft matrix theory and its decision making. *Computers and Mathematics with Applications, 59*(10), 3308-3314. https://doi.org/10.1016/j.camwa.2010.03.015

[13] Gong, X., Xiao, Z., & Zhang, X. (2010). The bijective soft set with its operations. *Computers and Mathematics with Applications, 60*(8), 2270-2278. https://doi.org/10.1016/j.camwa.2010.08.017

[14] Xiao, Z., Gong, K., Xia, S., & Zou, Y. (2010). Exclusive disjunctive soft sets. *Computers and Mathematics with Applications, 59*(6), 2128-2137. https://doi.org/10.1016/j.camwa.2009.12.018

[15] Feng, F., Li, Y., & Çağman, N. (2012). Generalized uni-int decision making schemes based on choice value soft sets. *European Journal of Operational Research, 220*(1), 162-170. https://doi.org/10.1016/j.ejor.2012.01.015

[16] Feng, Q., & Zhou, Y. (2014). Soft discernibility matrix and its applications in decision making. *Applied Soft Computing, 24*, 749-756. https://doi.org/10.1016/j.asoc.2014.08.042

[17] Kharal, A. (2014). Soft approximations and uni-int decision making. *The Scientific World Journal, 4*, 327408. https://doi.org/10.1155/2014/327408

[18] Dauda, M. K., Mamat, M., & Waziri, M. Y. (2015). An application of soft set in decision making. *Jurnal Teknologi, 77*(13), 119-122. https://doi.org/10.11113/jt.v77.6367

[19] Inthumathi, V., Chitra, V., & Jayasree, S. (2017). The role of operators on soft set in decision making problems. *International Journal of Computational and Applied Mathematics, 12*(3), 899-910.

[20] Atagün, A. O., Kamacı, H., & Oktay, O. (2018). Reduced soft matrices and generalized products with applications in decision making. *Neural Computing and Applications, 29*, 445-456. https://doi.org/10.1007/s00521-016-2542-y

[21] Kamacı, H., Saltık, K., Akız, H. F., & Atagün, A. O. (2018). Cardinality inverse soft matrix theory and its applications in multicriteria group decision making. *Journal of Intelligent & Fuzzy Systems, 34*(3), 2031-2049. https://doi.org/10.3233/JIFS-17876

[22] Yang, J. L., & Yao, Y. Y. (2020). Semantics of soft sets and three-way decision with soft sets. *Knowledge-Based Systems, 194*, 105538. https://doi.org/10.1016/j.knosys.2020.105538

[23] Petchimuthu, S., Garg, H., Kamacı, H., & Atagün, A. O. (2020). The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM. *Computational and Applied Mathematics, 39*(2), 1-32. https://doi.org/10.1007/s40314-020-1083-2

[24] Zorlutuna, İ. (2021). Soft set-valued mappings and their application in decision making problems. *Filomat, 35*(5), 1725-1733. https://doi.org/10.2298/FIL2105725Z

[25] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Computers and Mathematics with Applications, 45*(1), 555-562. https://doi.org/10.1016/S0898-1221(03)00016-6

[26] Pei, D., & Miao, D. (2005). From soft sets to information systems. In X. Hu, Q. Liu, A. Skowron, T. Y. Lin, R. R. Yager, & B. Zhang (Eds.), *Proceedings of Granular Computing* (2) (pp. 617-621). IEEE. https://doi.org/10.1109/GRC.2005.1547365

[27] Ali, M. I., Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Computers and Mathematics with Applications, 57*(9), 1547-1553. https://doi.org/10.1016/j.camwa.2008.11.009

[28] Yang, C. F. (2008). A note on: "Soft set theory" [*Computers & Mathematics with Applications* 45 (2003), 4-5, 555-562]. *Computers and Mathematics with Applications, 56*(7), 1899-1900. https://doi.org/10.1016/j.camwa.2008.03.019

[29] Feng, F., Li, Y. M., Davvaz, B., & Ali, M. I. (2010). Soft sets combined with fuzzy sets and rough sets: a tentative approach. *Soft Computing, 14*, 899-911. https://doi.org/10.1007/s00500-009-0465-6

[30] Jiang, Y., Tang, Y., Chen, Q., Wang, J., & Tang, S. (2010). Extending soft sets with description logics. *Computers and Mathematics with Applications, 59*(6), 2087-2096. https://doi.org/10.1016/j.camwa.2009.12.014

[31] Ali, M. I., Shabir, M., & Naz, M. (2011). Algebraic structures of soft sets associated with new operations. *Computers and Mathematics with Applications, 61*(9), 2647-2654. https://doi.org/10.1016/j.camwa.2011.03.011

[32] Yang, C. F. (2008). A note on soft set theory. *Computers and Mathematics with Applications, 56*(7), 1899-1900. https://doi.org/10.1016/j.camwa.2008.03.019

[33] Neog, I. J., & Sut, D. K. (2011). A new approach to the theory of soft set. *International Journal of Computer Applications, 32*(2), 1-6.

[34] Fu, L. (2011). Notes on soft set operations. *ARPN Journal of Systems and Softwares, 1*(6), 205-208.

[35] Ge, X., & Yang, S. (2011). Investigations on some operations of soft sets. *World Academy of Science, Engineering and Technology International Journal of Mathematical and Computational Sciences, 5*(3), 370-373.

[36] Singh, D., & Onyeozili, I. A. (2012). Some conceptual misunderstanding of the fundamentals of soft set theory. *ARPN Journal of Systems and Softwares, 2*(9), 251-254.

[37] Singh, D., & Onyeozili, I. A. (2012). Some results on distributive and absorption properties on soft operations. *IOSR Journal of Mathematics*, *4*(2), 18-30. https://doi.org/10.9790/5728-0421830

[39] Singh, D., & Onyeozili, I. A. (2012). Notes on soft matrices operations. *ARPN Journal of Science and Technology, 2*(9), 861-869.

[40] Zhu, P., & Wen, Q. (2013). Operations on soft sets revisited. *Journal of Applied Mathematics, 2013*, 1-7. https://doi.org/10.1155/2013/105752

[41] Sen, J. (2014). On algebraic structure of soft sets. *Annals of Fuzzy Mathematics and Informatics, 7*(6), 1013-1020.

[42] Eren, Ö. F. (2019). *On operations of soft sets*. Master's Thesis, Ondokuz Mayıs University, Samsun.

[43] Stojanovic, N. S. (2021). A new operation on soft sets: extended symmetric difference of soft sets. *Military Technical Courier, 69*(4), 779-791. https://doi.org/10.5937/vojtehg69-33655

[44] Sezgin, A., & Yavuz, E. (2023). A new soft set operation: Soft binary piecewise symmetric difference operation. *Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi, 5*(2), 189-208. https://doi.org/10.47112/neufmbd.2023.18

[45] Sezgin, A., & Sarıalioğlu, M. (2024). A new soft set operation: Complementary soft binary piecewise theta (θ) operation. *Journal of Kadirli Faculty of Applied Sciences, 4*(2), 325-357.

[46] Sezgin, A., & Çağman, N. (2024). A new soft set operation: Complementary soft binary piecewise difference operation. *Osmaniye Korkut Ata University Journal of the Institute of Science and Technology, 7*(1), 58-94. https://doi.org/10.47495/okufbed.1308379

[47] Sezgin, A., Aybek, F. N., & Güngör, N. B. (2023). A new soft set operation: Complementary soft binary piecewise union operation. *Acta Informatica Malaysia, 7*(1), 38-53. https://doi.org/10.26480/aim.01.2023.38.53

[48] Sezgin, A., Aybek, F. N., & Atagün, A. O. (2023). A new soft set operation: Complementary soft binary piecewise intersection operation. *Black Sea Journal of Engineering and Science, 6*(4), 330-346. https://doi.org/10.34248/bsengineering.1319873

[49] Sezgin, A., & Demirci, A. M. (2023). A new soft set operation: Complementary soft binary piecewise star operation. *Ikonion Journal of Mathematics, 5*(2), 24-52. https://doi.org/10.54286/ikjm.1304566

[50] Qin, K. Y., & Hong, Z. Y. (2010). On soft equality. *Journal of Computational and Applied Mathematics, 234*(5), 1347-1355. https://doi.org/10.1016/j.cam.2010.02.028

[51] Jun, Y. B., & Yang, X. (2011). A note on the paper "Combination of interval-valued fuzzy set and soft set" [*Comput. Math. Appl.* 58 (2009), 521-527]. *Computers and Mathematics with Applications, 61*(5), 1468-1470. https://doi.org/10.1016/j.camwa.2010.12.077

[52] Liu, X. Y., Feng, F. F., & Jun, Y. B. (2012). A note on generalized soft equal relations. *Computers and Mathematics with Applications, 64*(4), 572-578. https://doi.org/10.1016/j.camwa.2011.12.052

[53] Feng, F., & Yongming, L. (2013). Soft subsets and soft product operations. *Information Sciences, 232*, 44-57. https://doi.org/10.1016/j.ins.2013.01.001

[54] Abbas, M., Ali, B., & Romaguer, S. (2014). On generalized soft equality and soft lattice structure. *Filomat, 28*(6), 1191-1203. https://doi.org/10.2298/FIL1406191A

[55] Abbas, M., Ali, M. I., & Romaguera, S. (2017). Generalized operations in soft set theory via relaxed conditions on parameters. *Filomat, 31*(19), 5955-5964. https://doi.org/10.2298/FIL1719955A

[56] Alshami, T. (2019). Investigation and corrigendum to some results related to g-soft equality and g f-soft equality relations. *Filomat, 33*(11), 3375-3383. https://doi.org/10.2298/FIL1911375A

[57] Alshami, T., & El-Shafei, M. (2020). T-soft equality relation. *Turkish Journal of Mathematics, 44*(4), 1427-1441. https://doi.org/10.3906/mat-2005-117

[58] Ali, B., Saleem, N., Sundus, N., Khaleeq, S., Saeed, M., & George, R. A. (2022). Contribution to the theory of soft sets via generalized relaxed operations. *Mathematics, 10*(15), 26-36. https://doi.org/10.3390/math10152636

[59] Sezgin, A., Atagün, A. O., & Çağman, N. (in press). A complete study on and-product of soft sets. *Sigma Journal of Engineering and Natural Sciences*.

[60] Çağman, N. (2021). Conditional complements of sets and their application to group theory. *Journal of New Results in Science, 10*(3), 67-74. https://doi.org/10.54187/jnrs.1003890

[61] Sezgin, A., Çağman, N., Atagün, A. O., & Aybek, F. N. (2023). Complemental binary operations of sets and their application to group theory. *Matrix Science Mathematic, 7*(2), 91-98. https://doi.org/10.26480/msmk.02.2023.114.121

[62] Sezgin, A., & Sarıalioğlu, M. (2024). Complementary extended gamma operation: A new soft set operation. *Natural and Applied Sciences Journal, 7*(1), 15-44. https://doi.org/10.38061/idunas.1482044

[63] Sezgin, A., & Aybek, F. N. (2024). New restricted and extended soft set operations: Restricted gamma and extended gamma operations. *Big Data and Computing Visions, 4*(4), 272-306.

[64] Sezgin, A., Yavuz, E., & Atagün, A. O. (2024). Comprehension of soft binary piecewise gamma operation: A new operation for soft sets. *Journal of Advanced Mathematics and Mathematics Education, 7*(3), 13-40.

[65] Sezgin, A., & Aybek, F. N. (2024). A new soft set operation: Complementary soft binary piecewise gamma ( ) operation. *Matrix Science Mathematic, 7*(1), 27-45. https://doi.org/10.26480/msmk.01.2023.27.45

[66] Sezer, A. S., Atagün, A. O., & Çağman, N. (2013). A new view to N-group theory: Soft N-groups. *Fasciculi Mathematici, 51*, 123-140.

[67] Sezer, A. S. (2014). Certain characterizations of LA-semigroups by soft sets. *Journal of Intelligent and Fuzzy Systems, 27*(2), 1035-1046. https://doi.org/10.3233/IFS-131064

[68] Sezer, A. S. (2014). A new approach to LA-semigroup theory via the soft sets. *Journal of Intelligent and Fuzzy Systems, 26*(5), 2483-2495. https://doi.org/10.3233/IFS-130918

[69] Sezer, A. S., Çağman, N., & Atagün, A. O. (2014). Soft intersection interior ideals, quasi-ideals and generalized bi-ideals: A new approach to semigroup theory II. *Journal of Multiple-Valued Logic and Soft Computing, 23*(1-2), 161-207.

[70] Sezgin, A. (2016). A new approach to semigroup theory I: Soft union semigroups, ideals and bi-ideals. *Algebra Letters, 2016*(3), 1-46.

[71] Tunçay, M., & Sezgin, A. (2016). Soft union ring and its applications to ring theory. *International Journal of Computer Applications, 151*(9), 7-13. https://doi.org/10.5120/ijca2016911867

[72] Muştuoğlu, E., Sezgin, A., & Türk, Z. K. (2016). Some characterizations on soft uni-groups and normal soft uni-groups. *International Journal of Computer Applications, 155*(10), 1-8. https://doi.org/10.5120/ijca2016912412

[73] Khan, A., Izhar, M., & Sezgin, A. (2017). Characterizations of Abel Grassmann's groupoids by the properties of double-framed soft ideals. *International Journal of Analysis and Applications, 15*(1), 62-74.

[74] Sezgin, A., Çağman, N., & Atagün, A. O. (2017). A completely new view to soft intersection rings via soft uni-int product. *Applied Soft Computing, 54*, 366-392. https://doi.org/10.1016/j.asoc.2016.10.004

[75] Sezgin, A. (2018). A new view on AG-groupoid theory via soft sets for uncertainty modeling. *Filomat, 32*(8), 2995-3030. https://doi.org/10.2298/FIL1808995S

[76] Atagün, A. O., & Sezgin, A. (2018). Soft subnear-rings, soft ideals and soft N-subgroups of near-rings. *Mathematical Sciences Letters, 7*(1), 37-42. https://doi.org/10.18576/msl/070106

[77] Gulistan, M., Feng, F., Khan, M., & Sezgin, A. (2018). Characterizations of right weakly regular semigroups in terms of generalized cubic soft sets. *Mathematics, 6*, 293. https://doi.org/10.3390/math6120293

[78] Mahmood, T., Rehman, Z. U., & Sezgin, A. (2018). Lattice ordered soft near rings. *Korean Journal of Mathematics, 26*(3), 503-517.

[79] Jana, C., Pal, M., Karaaslan, F., & Sezgin, A. (2019). (α, β)-Soft intersectional rings and ideals with their applications. *New Mathematics and Natural Computation, 15*(2), 333-350. https://doi.org/10.1142/S1793005719500182

[80] Atagün, A. O., Kamacı, H., Taştekin, İ., & Sezgin, A. (2019). P-properties in near-rings. Journal of Mathematical and Fundamental Sciences, 51(2), 152-167. https://doi.org/10.5614/j.math.fund.sci.2019.51.2.5

[81] Özlü, Ş., & Sezgin, A. (2020). Soft covered ideals in semigroups. Acta Universitatis Sapientiae, *Mathematica, 10*(2), 317-346. https://doi.org/10.2478/ausm-2020-0023

[82] Sezgin, A., Atagün, A. O., Çağman, N., & Demir, H. (2022). On near-rings with soft union ideals and applications. *New Mathematics and Natural Computation, 18*(2), 495-511. https://doi.org/10.1142/S1793005722500247

[83] Manikantan, T., Ramasany, P., & Sezgin, A. (2023). Soft quasi-ideals of soft near-rings. *Sigma Journal of Engineering and Natural Science, 41*(3), 565-574. https://doi.org/10.14744/sigma.2023.00062

[84] Sezer, A. S., Atagün, A. O., & Çağman, N. (2014). N-group SI-action and its applications to N-Group Theory. *Fasciculi Mathematici, 52*, 139-153.

[85] Sezer, A. S., Çağman, N., & Atagün, A. O. (2015). Uni-soft substructures of groups. *Annals of Fuzzy Mathematics and Informatics, 9*(2), 235-246.

[86] Atagün, A. O., & Sezer, A. S. (2015). Soft sets, soft semimodules and soft substructures of semimodules. *Mathematical Sciences Letters, 4*(3), 235-242. http://dx.doi.org/10.12785/msl/040303

[87] Atagün, A. O., & Sezgin, A. (2017). Int-soft substructures of groups and semirings with applications. *Applied Mathematics & Information Sciences, 11*(1), 105-113. https://doi.org/10.18576/amis/110113

[88] Sezer, A. S., & Atagün, A. O. (2016). A new kind of vector space: Soft vector space. *Southeast Asian Bulletin of Mathematics, 40*, 753-770. http://www.seams-bull-math.ynu.edu.cn/archive.jsp

[89] Atagün, A. O., & Sezgin, A. (2018). A new view to near-ring theory: Soft near-rings. *Southeast Asian Journal of Mathematics & Mathematical Sciences, 14*(3), 1-14.

[90] Atagün, A. O., & Sezgin, A. (2022). More on prime, maximal and principal soft ideals of soft rings. *New Mathematics and Natural Computation, 18*(1), 195-207. https://doi.org/10.1142/S1793005722500119

[91] Riaz, M., Hashmi, M. R., Karaaslan, F., Sezgin, A., Shamiri, M. M. A. A., & Khalaf, M. M. (2023). Emerging trends in social networking systems and generation gap with neutrosophic crisp soft mapping. *Computer Modeling in Engineering & Sciences, 136*, 1759-1783. https://doi.org/10.32604/cmes.2023.023327

[92] Sezgin, A., & Çam, N. H. (2024). Soft plus-product: A new product for soft sets with its decision-making. *Complexity Analysis and Applications*. https://caa.reapress.com/journal/article/view/33 (in press).

[93] Gulistan, M., & Hassan, N. (2019). A generalized approach towards soft expert sets via neutrosophic cubic sets with applications in games. *Symmetry, 11*(2), 289. https://doi.org/10.3390/sym11020289

[94] Memiş, S. (2022). Another view on picture fuzzy soft sets and their product operations with soft decision-making. *Journal of New Theory*, (*38*), 1-13. https://doi.org/10.53570/jnt.1037280