Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 2, 2025, Pages 201-209
https://doi.org/10.34198/ejms.15225.201209

On Extension of Existing Results on the Diophantine Equation: $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$

Nyakebogo Abraham Osogo[1,*] and Kimtai Boaz Simatwo[2]

[1] Department of Mathematics and Actuarial Science, Kisii University, P. O. Box 408-40200, Kisii, Kenya
e-mail: abrahampsogo@kisiiuniversity.ac.ke

[2] Department of Mathematics, Masinde Muliro University of Science and Technology, P.O.Box 190-50100, Kakamega, Kenya
e-mail: kimtaiboaz96@gmail.com

Abstract

Let w_r be a given sequence in arithmetic progression with common difference d. The study of diophantine equation, which are polynomial equations seeking integer solutions has been a very interesting journey in the field of number theory. Historically, these equations have attracted the attention of many mathematicians due to their intrinsic challenges and their significance in understanding the properties of integers. In this current study we examine a diophantine equation relating the sum of square integers from specific sequences to a variable d. In particular, on extension of existing results on the diophantine equation: $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{n=1}^{\frac{n}{3}} w_{3r-1}^2)$ is introduced and partially characterized.

1 Introduction

Diophantine equations, tracing their roots back to the error of ancient Greek mathematician Diophantus, continue to be a captivating challenge within number theory. These equations seeking integers solutions, hold significant importance due to their real life applications. Despite the extensive exploration of various diophantine equation, including renowned challenges like Fermat's Last Theorem, Ramanugn. Nagell equation and Lebesque Nagell, as well as studies focusing of polynomials of degree less than five, the specific examinations of the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{n=1}^{\frac{n}{3}} \sum_{3r-1}^{2})$ remains largely uncharted. Recent research has delved into the intricacies of polynomials with degrees less than five as referenced [1,3,5,9,13,15] for a comprehensive understanding of studies related to Fermat's Last Theorem and Baranujan Nagell equations readers are encouraged to explore [2,4,7,8,10–12,14,16] within the existing

Received: October 29, 2024; Revised & Accepted: December 4, 2025; Published: January 5, 2025
2020 Mathematics Subject Classification: 11D61.
Keywords and phrases: sequences, diophantine equation, integer, polynomial, factorization.
*Corresponding author

body of work the literature concerning the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{n=1}^{\frac{n}{3}} \sum_{3r-1}^{2})$ remains largely unexplored. This study aims to contribute to this knowledge gap on extension of existing results on the Diophantine Equation: $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$ which was first introduced by Mude *et al*. in [12] and Najman in [14], thus seeking to enhance our comprehension of this specific diophantine equation within the broader landscape of mathematical exploration.

2 Main Results

Theorem 1.1: *Consider the condition satisfying the equation* $(n, w_1, w_2, ..., w_{15}, 5d) = (15, w_1, w_2, ..., w_{15}, 5d)$.

Then, the diophantine equation:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + 5d^2 = 3(5d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2)$$

has the solution in integers if $w_{15} - w_{14} = w_{14} - w_{13} = w_{13} - w_{12} = w_{12} - w_{11} = w_{11} - w_{10} = w_{10} - w_9 = w_9 - w_8 = w_8 - w_7 = w_7 - w_6 = w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d$.

Proof: Consider the equation

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + 5d^2 = 3(5d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2).$$

And suppose that $w_2 = w_1 + d, w_3 = w_1 + 2d, w_4 = w_1 + 3d, w_5 = w_1 + 4d, w_6 = w_1 + 5d, w_7 = w_1 + 6d, w_8 = w_1 + 7d, w_9 = w_1 + 8d, w_{10} = w_1 + 9d, w_{11} = w_1 + 10d, w_{12} = w_1 + 11d, w_{13} = w_1 + 12d, w_{14} = w_1 + 13d, w_{15} = w_1 + 14d$.

Hence:

And suppose that $w_1^2 + (w_1 + d)^2 + (w_1 + 2d)^2 + (w_1 + 3d)^2 + (w_1 + 4d)^2 + (w_1 + 5d)^2 + (w_1 + 6d)^2 + (w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+5d^2$.

Simplifies to

$15w_1^2 + 210w_1d + 1020d^2 = 3(5w_1^2 + 70w_1d + 340d^2)$. ...(1.1)

Splitting equation (1.1) into thrice sums of squares, we obtain:

$3(5d^2 + (w_1^2 + 2w_1d + d^2) + (w_1^2 + 8w_1d + 16d^2) + (w_1^2 + 14w_1d + 49d^2) + (w_1^2 + 20w_1d + 100d^2) + (w_1^2 + 26w_1d + 169d^2))$

$= 3(5d^2) + (w_1 + d)^2 + (w_1 + 4d)^2 + (w_1 + 7d)^2 + (w_1 + 10d)^2$

$= 3(5d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2)$.

This completes the proof. □

Theorem 1.2: *Consider the condition satisfying the equation* $(n, w_1, w_2, ..., w_{18}, 6d) = (15, w_1, w_2, ..., w_{18}, 6d)$.

Then, the diophantine equation:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + w_{16}^2 + w_{17}^2 + w_{18}^2 + 6d^2$$
$$= 3(6d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2)$$

has the solution in integers if $w_{18} - w_{17} = w_{17} - w_{16} = w_{16} - w_{15} = w_{15} - w_{14} = w_{14} - w_{13} = w_{13} - w_{12} = w_{12} - w_{11} = w_{11} - w_{10} = w_{10} - w_9 = w_9 - w_8 = w_8 - w_7 = w_7 - w_6 = w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d$.

Proof: Consider the equation

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+6d^2$
$= 3(6d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2)$.

And suppose that $w_2 = w_1+d, w_3 = w_1+2d, w_4 = w_1+3d, w_5 = w_1+4d, w_6 = w_1+5d, w_7 = w_1+6d, w_8 = w_1+7d, w_9 = w_1+8d, w_{10} = w_1+9d, w_{11} = w_1+10d, w_{12} = w_1+11d, w_{13} = w_1+12d, w_{14} = w_1+13d, w_{15} = w_1+14d, w_{16} = w_1+15d, w_{17} = w_1+16d, w_{18} = w_1+17d$.

Hence:

And suppose that $w_1^2+(w_1+d)^2+(w_1+2d)^2+(w_1+3d)^2+(w_1+4d)^2+(w_1+5d)^2+(w_1+6d)^2+(w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+6d^2$.

Simplifies to

$18w_1^2+306w_1d+1791d^2 = 3(6w_1^2+102w_1d+597d^2)$. ...(1.2)

Splitting equation (1.2) into thrice sums of squares, we obtain:

$3(6d^2+(w_1^2+2w_1d+d^2)+(w_1^2+8w_1d+16d^2)+(w_1^2+14w_1d+49d^2)+(w_1^2+20w_1d+100d^2)+(w_1^2+26w_1d+169d^2+(w_1^2+32w_1d+256d^2))$

$= 3(6d^2)+(w_1+d)^2+(w_1+4d)^2+(w_1+7d)^2+(w_1+10d)^2+(w_1+13d)^2+(w_1+16d)^2$

$= 3(6d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2)$.

This completes the proof. □

Theorem 1.3: *Consider the condition satisfying the equation* $(n, w_1, w_2, ..., w_{21}, 7d) = (21, w_1, w_2, ..., w_{21}, 7d)$.

Then, the diophantine equation:

$$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+7d^2$$
$$=3(7d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2)$$

has the solution in integers if $w_{21}-w_{20}=w_{20}-w_{19}=w_{19}-w_{18}=w_{18}-w_{17}=w_{17}-w_{16}=w_{16}-w_{15}=w_{15}-w_{14}=w_{14}-w_{13}=w_{13}-w_{12}=w_{12}-w_{11}=w_{11}-w_{10}=w_{10}-w_9=w_9-w_8=w_8-w_7=w_7-w_6=w_6-w_5=w_5-w_4=w_4-w_3=w_3-w_2=w_2-w_1=d.$

Proof: Consider the equation

$$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+7d^2$$
$$=3(7d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2).$$

And suppose that $w_2=w_1+d, w_3=w_1+2d, w_4=w_1+3d, w_5=w_1+4d, w_6=w_1+5d, w_7=w_1+6d, w_8=w_1+7d, w_9=w_1+8d, w_{10}=w_1+9d, w_{11}=w_1+10d, w_{12}=w_1+11d, w_{13}=w_1+12d, w_{14}=w_1+13d, w_{15}=w_1+14d, w_{16}=w_1+15d, w_{17}=w_1+16d, w_{18}=w_1+17d, w_{19}=w_1+18d, w_{20}=w_1+19d, w_{21}=w_1+20d.$

Hence:

And suppose that $w_1^2+(w_1+d)^2+(w_1+2d)^2+(w_1+3d)^2+(w_1+4d)^2+(w_1+5d)^2+(w_1+6d)^2+(w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+(w_1+18d)^2+(w_1+19d)^2+(w_1+20d)^2+7d^2.$

Simplifies to

$$21w_1^2+420w_1d+2877d^2=3(7w_1^2+140w_1d+959d^2). \quad ...(1.3)$$

Splitting equation (1.3) into thrice sums of squares, we obtain:

$3(7d^2 + (w_1^2 + 2w_1d + d^2) + (w_1^2 + 8w_1d + 16d^2) + (w_1^2 + 14w_1d + 49d^2) + (w_1^2 + 20w_1d + 100d^2) + (w_1^2 + 26w_1d + 169d^2 + (w_1^2 + 32w_1d + 256d^2) + (w_1^2 + 38w_1d + 361d^2))$

$= 3(7d^2) + (w_1 + d)^2 + (w_1 + 4d)^2 + (w_1 + 7d)^2 + (w_1 + 10d)^2 + (w_1 + 13d)^2 + (w_1 + 16d)^2 + (w_1 + 19d)^2$

$= 3(7d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2 + w_{20}^2)$.

This completes the proof. □

Theorem 1.4: *Consider the condition satisfying the equation* $(n, w_1, w_2, ..., w_{24}, 7d) = (24, w_1, w_2, ..., w_{24}, 8d)$.

Then, the diophantine equation:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + w_{16}^2 + w_{17}^2 + w_{18}^2 + w_{19}^2 + w_{20}^2 + w_{21}^2 + w_{22}^2 + w_{23}^2 + w_{24}^2 + 8d^2$$
$$= 3(8d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2 + w_{20}^2 + w_{23}^2)$$

has the solution in integers if $w_{23} - w_{22} = w_{22} - w_{21} = w_{21} - w_{20} = w_{20} - w_{19} = w_{19} - w_{18} = w_{18} - w_{17} = w_{17} - w_{16} = w_{16} - w_{15} = w_{15} - w_{14} = w_{14} - w_{13} = w_{13} - w_{12} = w_{12} - w_{11} = w_{11} - w_{10} = w_{10} - w_9 = w_9 - w_8 = w_8 - w_7 = w_7 - w_6 = w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d$.

Proof: Consider the equation

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + w_{16}^2 + w_{17}^2 + w_{18}^2 + w_{19}^2 + w_{20}^2 + w_{21}^2 + w_{22}^2 + w_{23}^2 + w_{24}^2 + 8d^2$$
$$= 3(8d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2 + w_{20}^2 + w_{23}^2).$$

And suppose that $w_2 = w_1 + d, w_3 = w_1 + 2d, w_4 = w_1 + 3d, w_5 = w_1 + 4d, w_6 = w_1 + 5d, w_7 =$

$w_1 + 6d, w_8 = w_1 + 7d, w_9 = w_1 + 8d, w_{10} = w_1 + 9d, w_{11} = w_1 + 10d, w_{12} = w_1 + 11d, w_{13} = w_1 + 12d, w_{14} = w_1 + 13d, w_{15} = w_1 + 14d, w_{16} = w_1 + 15d, w_{17} = w_1 + 16d, w_{18} = w_1 + 17d, w_{19} = w_1 + 18d, w_{20} = w_1 + 19d, w_{21} = w_1 + 20d, w_{22} = w_1 + 21d w_{23} = w_1 + 22d w_{24} = w_1 + 23d.$

Hence:

And suppose that $w_1^2+(w_1+d)^2+(w_1+2d)^2+(w_1+3d)^2+(w_1+4d)^2+(w_1+5d)^2+(w_1+6d)^2+(w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+(w_1+18d)^2+(w_1+19d)^2+(w_1+20d)^2+(w_1+21d)^2+(w_1+22d)^2+(w_1+23d)^2+8d^2.$

Simplifies to

$$24w_1^2 + 552w_1d + 4332d^2 = 3(8w_1^2 + 184w_1d + 1444d^2). \ ...(1.4)$$

Splitting equation (1.3) into thrice sums of squares, we obtain:

$$3(8d^2 + (w_1^2 + 2w_1d + d^2) + (w_1^2 + 8w_1d + 16d^2) + (w_1^2 + 14w_1d + 49d^2) + (w_1^2 + 20w_1d + 100d^2) + (w_1^2 + 26w_1d + 169d^2 + (w_1^2 + 32w_1d + 256d^2) + (w_1^2 + 38w_1d + 361d^2) + (w_1^2 + 44w_1d + 484d^2))$$

$$= 3(8d^2)+(w_1+d)^2+(w_1+4d)^2+(w_1+7d)^2+(w_1+10d)^2+(w_1+13d)^2+(w_1+16d)^2+(w_1+19d)^2+(w_1+22d)^2$$

$$= 3(7d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2 + w_{20}^2 + w_{23}^2).$$

This completes the proof. □

3 Conclusion

In summary, the solution of the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{n=1}^{\frac{n}{3}} w_{3r-1}^2)$, under the specified conditions of a common difference d between consecutive terms $w_n, w_{n-1}, ..., w_2, w_1$ where $w_n - w_{n-1} = w_{n-1} - w_{n-2} = ... = w_2 - w_1 = d$ has been achieved for some cases. This solution provides valuable insights into the relation among the sequence terms, enhancing our understanding of the inherent patterns and structures within the equation. For future investigations, it is recommended to explore extensions of this diophantine equation by proving conjecture (1).

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Acknowledgements

The authors would like to thank the anonymous reviewers for carefully reading the article and for their helpful comments.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Bombieri, E., & Bourgain, J. (2015). A problem on sums of two squares. *International Mathematics Research Notices, 2015*(11), 3343-3407. https://doi.org/10.1093/imrn/rnu005

[2] Cai, Y. (2016). Waring-Goldbach problem: two squares and higher powers. *Journal de Théorie des Nombres de Bordeaux, 28*(3), 791-810. https://doi.org/10.5802/jtnb.964

[3] Cavallo, A. (2019). An elementary computation of the Galois groups of symmetric sextic trinomials. *arXiv preprint arXiv:1902.00965*. https://doi.org/10.48550/arXiv.1902.00965

[4] Christopher, A. D. (2016). A partition-theoretic proof of Fermat's two squares theorem. *Discrete Mathematics, 339*(4), 1410-1411. https://doi.org/10.1016/j.disc.2015.12.002

[5] Fathi, A., Mobadersany, P., & Fathi, R. (2012). A simple method to solve quartic equations. *Australian Journal of Basic and Applied Sciences, 6*(6), 331-336.

[6] Kimtai, B. S., & Mude, L. H. (2023). On generalized sums of six, seven, and nine cubes. *Science Mundi, 3*(1), 135-142. https://doi.org/10.51867/scimundi.3.1.14

[7] Kouropoulos, G. P. (2021). A combined methodology for the approximate estimation of the roots of the general sextic polynomial equation. *Unpublished*. https://doi.org/10.21203/rs.3.rs-882192/v1

[8] Mude, L. H., Kayiita, Z. K., & Ndung'u, K. J. (2023). Some generalized formula for sums of cube. *Journal of Advances in Mathematics and Computer Science, 38*(8), 47-52. https://doi.org/10.9734/jamcs/2023/v38i81789

[9] Mochimaru, Y. (2005). Solution of sextic equations. *International Journal of Pure and Applied Mathematics, 23*(4), 577. Academic Publications.

[10] Mude, L. H. (2022). Some formulae for integer sums of two squares. *Journal of Advances in Mathematics and Computer Science, 37*(4), 53-57. https://doi.org/10.9734/jamcs/2022/v37i430448

[11] Mude, L. H. (2024). On some mixed polynomial exponential Diophantine equation: $(\alpha^n+\beta^n+a(\alpha^s\pm\beta^s)^m+D = r(u^k+v^k+w^k))$ with (α) and (β) consecutive. *Journal of Advances in Mathematics and Computer Science, 39*(10), 11-17. https://doi.org/10.9734/jamcs/2024/v39i101931

[12] Mude, L. H., Ndung'u, K. J., & Kayiita, Z. K. (2024). On sums of squares involving integer sequence: $\left(\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{n} \frac{n}{3} w_{3r-1}^2\right)\right)$. *Journal of Advances in Mathematics and Computer Science, 39*(7), 1-6. https://doi.org/10.9734/jamcs/2024/v39i71906

[13] Najman, F. (2010a). The Diophantine equation $x^4 \pm y^4 = iz^2$ in Gaussian integers. *The American Mathematical Monthly, 117*(7), 637-641. https://doi.org/10.4169/000298910x496769

[14] Najman, F. (2010b). Torsion of elliptic curves over quadratic cyclotomic fields. *arXiv preprint arXiv:1005.0558*. https://doi.org/10.48550/arXiv.1005.0558

[15] Simatwo, K. B. (2024). Equal sums of four even powers. *Asian Research Journal of Mathematics, 20*(5), 50-54. https://doi.org/10.9734/arjom/2024/v20i5802

[16] Tignol, J.-P. (2015). *Galois' theory of algebraic equations*. World Scientific Publishing Company. https://doi.org/10.1142/9719