

# Variational Iteration Method for the Solution of Differential Equation of Motion of the Mathematical Pendulum and Duffing-Harmonic Oscillator

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## Abstract

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In this work, the differential equation of motion of the undamped mathematical pendulum and Duffing-harmonic oscillator are discussed by using the variational iteration method. Additionally, common problems of pendulum are classified and Lagrange multipliers are obtained for each type of problem. Examples are given for illustration.

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## 1. Introduction

Vibration of dynamical systems can be divided into two main classes like *discrete* and *distributed*. The variables in discrete systems depend on time only, whereas in distributed systems such as beams, plates, etc. variables depend on time and space. Therefore, equations of motion of discrete systems are described by ordinary differential equations, while equations of motion of distributed systems are described by partial differential equations [1].

The variational iteration method [2-13] has been used to solve many nonlinear PDEs, ordinary differential equations such that wave solutions, rational solutions, compacton solutions and other types of solution were found by Abdou and Soliman [14]. Additionally, He [15] used VIM to solve linear/nonlinear vibration problems.

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The procedure presented in this paper can be simply extended to solve more complex vibration problems; such as aeroelasticity, random vibrations etc.

## 2. Variational Iteration Method

In order to illustrate the basic concepts of VIM, the following nonlinear partial differential equation can be considered

$$Ru(x, t) + Nu(x, t) = g(x, t), \quad (1)$$

where  $R$  is a linear operator which has partial derivatives with respect to  $Nu(x, t)$  is a nonlinear term and  $g(x, t)$  is an inhomogeneous term.

According to VIM, the following iteration formula can be constructed.

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda [R\tilde{u}_n + N\tilde{u}_n - g] d\tau, \quad (2)$$

where  $\lambda$  is the general Lagrange multiplier which can be identified optimally via variational theory,  $R\tilde{u}_n$  and  $N\tilde{u}_n$  are considered as restricted variations, i.e.,

$$\delta R\tilde{u}_n = 0, \quad \delta N\tilde{u}_n = 0.$$

## 3. Examples

### Example 1.

In this example, Mathematical Pendulum that was studied by He [15, 16] is considered.

The differential equation of motion of the undamped mathematical pendulum is given by

$$\ddot{y} + \omega^2 \sin y = 0. \quad (3)$$

The initial conditions for this problem are as follows:

$$y(0) = A, \quad (4a)$$

$$\dot{y}(0) = 0. \quad (4b)$$

The  $\sin y$  term in Eq. (3) is a nonlinear term and it can be expanded as

$$\sin y \approx y - \frac{1}{6} y^3. \quad (5)$$

Substituting Eq. (4) into Eq. (5) gives

$$\ddot{y} + \omega^2 y - \frac{\omega^2}{6} y^3 = 0. \quad (6)$$

A more detailed form of this mathematical pendulum was investigated by He [15, 16].

The Lagrange multiplier of this problem is

$$\lambda = \frac{1}{\omega} \sin[\omega(\tau - t)]. \quad (7)$$

Hence the iteration formula is

$$y_{n+1}(t) = y_n(t) + \frac{1}{\omega} \int_0^t \sin[\omega(\tau - t)] \left[ y''(\tau) + \omega^2 y(\tau) - \frac{\omega^2}{6} y^3(\tau) \right] d\tau. \quad (8)$$

The complementary solution of this problem that is used as an initial approximation is given by

$$y_0(t) = A \cos(\alpha \omega t), \quad (9)$$

where  $\alpha$  is an unknown constant.

Substituting the initial approximation into Eq. (5), the following residual is obtained

$$\begin{aligned} R_0(t) &\approx \ddot{y} + \omega^2 y - \frac{\omega^2}{6} y^3 \\ &= A \left( 1 - \frac{1}{8} A^2 - \alpha^2 \right) \omega^2 \cos(\alpha \omega t) - \frac{1}{24} A^3 \omega^2 \cos(3\alpha \omega t). \end{aligned} \quad (10)$$

The coefficient of the  $\cos(\alpha \omega t)$  term is set to zero in order to eliminate the secular term which may occur in the next iteration. Doing so, the expression of  $\alpha$  is found as follows

$$\alpha = \sqrt{1 - \frac{A^2}{8}}. \quad (11)$$

Hence,

$$y_1(t) = A \cos \alpha \omega t - \frac{A^3}{24(9\alpha^2 - 1)\omega^2} (\cos 3\alpha \omega t - \cos \omega t) \quad (12)$$

with  $\alpha$  defined in Eq. (11).

The period can be expressed as follows

$$T = \frac{2\pi}{\omega \sqrt{1 - \frac{1}{8} A^2}}. \quad (13)$$

If  $A = \frac{\pi}{2}$ , then  $T = 1.20T_0$ . On the other hand He's [15, 16] approximation gives  $T = 1.17T_0$ , while the exact period is  $T_{ex} = 1.16T_0$ , where  $T_0 = 2\pi/\omega$ .

### Example 2.

In this example, the problem that was studied by Nayfeh and Mook [17] is considered.

The differential equation of motion is given by,

$$\ddot{u} + \omega^2 u + \varepsilon u^2 \ddot{u} = 0. \quad (14)$$

The initial conditions for this problem are as follows:

$$y(0) = A, \quad (15a)$$

$$\dot{y}(0) = 0. \quad (15b)$$

The Lagrange multiplier of this problem is

$$\lambda = \frac{1}{\omega} \sin[\omega(\tau - t)]. \quad (16)$$

The iteration formula is given by

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega} \int_0^t \sin[\omega(\tau - t)] [u''(\tau) + \omega^2 u(\tau) + \varepsilon u^2(\tau) u''(\tau)] d\tau. \quad (17)$$

The complementary solution of this problem that is used as an initial approximation is given by

$$u_0(t) = A \cos(\alpha\omega t), \quad (18)$$

where  $\alpha$  is an unknown constant.

Substituting the initial approximation given by Eq. (18), the following residual is obtained as follows

$$\begin{aligned} R_0(t) &\approx \ddot{u} + \omega^2 u + \varepsilon u^2 \ddot{u} \\ &= A\omega^2 \left( 1 - \alpha^2 - \frac{3}{4} A^2 \varepsilon \right) \cos(\alpha\omega t) - \frac{1}{4} \varepsilon A^3 \alpha^2 \omega^2 \cos(3\alpha\omega t). \end{aligned} \quad (19)$$

The coefficient of the  $\cos(\alpha\omega t)$  term is set to zero in order to eliminate the secular term which may occur in the next iteration. Doing so, the expression for  $\alpha$  is obtained as follows

$$\alpha = \frac{2}{\sqrt{4 + 3\varepsilon A^2}}. \quad (20)$$

Hence,

$$y_1(t) = A \cos \alpha\omega t + \frac{\varepsilon A^3 \alpha^2}{4(9\alpha^2 - 1)} (\cos \omega t - \cos 3\alpha\omega t) \quad (21)$$

with  $\alpha$  defined in Eq. (20).

The new frequency is defined as follows

$$\omega_1 = \alpha\omega \Rightarrow \omega_1 = \frac{2}{\sqrt{4 + 3\varepsilon A^2}} \omega. \quad (22)$$

The frequency that is obtained by Nayfeh and Mook [17] using the perturbation method is

$$\omega_1 = \omega \left( 1 - \frac{3}{8} \varepsilon A^2 \right). \quad (23)$$

Note that Eq. (23) is valid only for small  $\varepsilon$  values. However, the frequency expression given by Eq. (22) is valid for all  $\varepsilon$  values and takes the following form for small  $\varepsilon$

values

$$\omega_1 \approx 1 - \frac{3}{8}\epsilon A^2 + \frac{27}{128}\epsilon^2 A^4 + \dots \quad (24)$$

### Example 3.

In this example, the Duffing-harmonic oscillator that was studied by Mickens [18] and Lim and Wu [19] is considered.

The differential equation of motion is given by,

$$\frac{d^2 y}{dt^2} + \frac{y^3}{1 + y^2} = 0. \quad (25)$$

The initial conditions for this problem are as follows:

$$y(0) = A, \quad (26a)$$

$$\dot{y}(0) = 0. \quad (26b)$$

For small  $y$  values, Eq. (25) reduces to

$$\frac{d^2 y}{dt^2} + y^3 \approx 0. \quad (27a)$$

On the other hand, for large  $y$  values, Eq. (25) reduces to

$$\frac{d^2 y}{dt^2} + y \approx 0. \quad (27b)$$

Considering Eqs. (26a) and (26b) respectively, it is noticed that for small  $y$  values, Eq. (27) reduces to the equation of motion of the Duffing-type nonlinear oscillator while for large  $y$  values, it reduces to the equation of motion of a linear harmonic oscillator. Therefore, Eq. (27) is called as Duffing-harmonic oscillator equation of motion.

The following form of Eq. (27) is going to be studied in this example

$$(1 + y^2) \frac{d^2 y}{dt^2} + y^3 = 0. \quad (28)$$

He's technique is going to be used to overcome secular terms that appear in the

iterations. The initial approximation is,

$$y_0(t) = A \cos(\alpha t) \quad (29)$$

where  $\alpha$  is an unknown constant.

Substituting the initial approximation into Eq. (28), the following residual is obtained

$$\begin{aligned} R_0(t) &\approx (1 + y^2) \ddot{y} + y^3 \\ &= \left( \frac{3}{4} A^2 - \alpha^2 - \frac{3}{4} A^2 \alpha^2 \right) \cos(\alpha t) + \frac{A^3}{4} (1 - \alpha^2) \cos(3\alpha t). \end{aligned} \quad (30)$$

In order to discard the secular terms, the coefficient of  $\cos(\alpha t)$  is set to zero which gives the expression of  $\alpha$  as follows

$$\alpha = \sqrt{\frac{\frac{3}{4} A^2}{1 + \frac{3}{4} A^2}}. \quad (31)$$

Hence the new frequency is defined as follows

$$\omega = \sqrt{\frac{\frac{3}{4} A^2}{1 + \frac{3}{4} A^2}} \quad (32)$$

which is the same with the one found by Mickens [18].

The iteration formula is given by

$$u_1(t) = u_0(t) + \int_0^t (\tau - 1) \left[ \frac{A^3}{4} (1 - \alpha^2) \cos(3\alpha\tau) \right] d\tau. \quad (33)$$

Hence,

$$y_1(t) = \cos \omega t + \frac{A}{27} (\cos 3\omega t - 1) \quad (34)$$

with  $\omega$  defined in Eq. (30).

For small values of amplitude  $A$ , the frequency expression given in Eq. (32) is expressed as follows

$$\omega \simeq \sqrt{\frac{3}{4}}A. \quad (35a)$$

Additionally, for large values of amplitude  $A$ , the frequency expression given in Eq. (32) is expressed as follows

$$\omega \simeq 1 \quad (35b)$$

which agree with the approximations made for the equations of motion.

#### 4. Conclusion

In this work, the differential equation of motion of the undamped mathematical pendulum and Duffing-harmonic oscillator are solved using the variational iteration method. Additionally, the procedure presented in this paper can be simply extended to solve more complex vibration problems; such as aeroelasticity, random vibrations etc.

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