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On the Solution of a Fractional-order Biological Population Model using q-Laplace Homotopy Analysis Method (qLHAM)

Oluwatope Richard Ojo

Department of Mathematics, East Tennessee State University, Johnson City, USA
e-mail: ojoor2@etsu.edu

Abstract

In this paper, we study a type of biological population model in its fractional order using the q-Laplace homotopy analysis method. This method, which combines the Laplace transform, q-calculus, and the homotopy analysis method developed by Shijun Liao in [11], is employed to provide approximate analytical solutions to the biological population model. Furthermore, we illustrate the dynamical behavior of this model graphically.

1 Introduction

Dynamical systems are mostly modeled using partial differential equations (PDEs), and many research work have focused on nonlinear PDEs in their integer and non-integer forms in recent years. The main reason for the use of non-integer differential equations for modeling is due to their widespread applications in different areas of science and engineering especially electrochemistry, acoustics, electromagnetic, viscoelasticity chemical processes, physics, material science, engineering, and biology [1, 2]. Fractional derivatives in differential equations allows for the consideration of a function's past behavior across a range of values instead of only its present state [1, 2].

Solving non-integer order differential equations can be a daunting task and this is why the past few years have also seen the advent of different methods for their solutions. [16–18], especially [18], contains list of methods that have been deployed over the years for non-integer differential equation and they include: Adomian Decomposition method, homotopy analysis method, homotopy perturbation method, and so on.

In this article, we will provide approximate analytical solutions of the biological population model with fractional order using q-Laplace-homotopy analysis (qLHAM) approach, where Laplace-Homotopy

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analysis method is a combination of Laplace transform and the homotopy analysis method. The concept of q-deformation plays a significant role in the study of dynamical systems, particularly in how symmetry can be disrupted. In these systems, the fundamental symmetry present in their equilibrium state may not hold, leading to intriguing behaviors and characteristics.

The q-deformed equations, characterized by the deformation parameter q, modify traditional algebraic structures, introducing a rich framework for analysis. These equations find applications in various physical contexts, enabling the description of particle behavior, field dynamics, and interactions where noncommutative geometry or quantum group symmetries are relevant. By manipulating the value of q, researchers can explore different regimes of a system, revealing insights into phenomena that may not be observable within the conventional frameworks. This approach has opened new avenues for understanding complex systems in areas such as quantum physics, statistical mechanics, and beyond.

2 Biological Population Models

The term population refers to people living within a political or geographical boundary, but in the biological sense, it is a collection of organisms of a particular species that share the same characteristics living in a given area. With this definition we can derive a simple population model considering birth and death within the area as:

$$r = (b - d) + (i - m), \tag{2.1}$$

where b is birthrate; d, death rate; m, movement out of area (*i.e.*, emmigration) and i is the net immigration. Assuming the population grows without limits at its maximal rate, the mathematical model is defined by:

$$\frac{dN}{dt} = r_i N, \tag{2.2}$$

where N is the total number of individuals in the population, $\frac{dN}{dt}$ is the rate of change of N over time t, and r_i is the innate capacity for growth. Solving (2.2) using separation of variables, we have

$$N = N_0 e^{r_i t}, \tag{2.3}$$

where N_0 is the initial population, and $e^{r_i t}$ is the exponential function defined by:

$$e^{r_i t} = \sum_{m=0}^{\infty} \frac{(r_i t)^m}{m!} = 1 + (r_i t) + \frac{(r_i t)^2}{2!} + \frac{(r_i t)^3}{3!} + \dots \tag{2.4}$$

It is not the scope of this work to work on which model best suit a population, our aim is to point out some parameters that are available in most population model and apply them to this work. Interested readers may find a lot of introductory text that touches more on this and extends to more realistic models like the Logistic model, Lotka-Volterra model, species-area relationship, and so on.

If we let γ and $\eta(\gamma)$ be functions of time t and position x, y in a region R of a biological specie in R with γ and η representing population density and population supply due to births and deaths respectively in the nonlinear biological population model [4]:

$$\frac{\partial^\alpha \gamma}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}(\gamma^2) + \frac{\partial^2}{\partial y^2}(\gamma^2) + \eta(\gamma), \qquad 0 < \alpha \leq 1,\ x, y \in \mathbb{R},\ t \geq 0 \tag{2.5}$$

with initial condition $\gamma(x, y, 0)$. When $\alpha \to 1$ in (2.5), the following are mathematical explanations of some biological processes for $\eta(\gamma)$:

1. $\eta(\gamma) = c$ leads to the Malthusian Law [19], where c is an arbitrary constant.
2. $\eta(\gamma) = \gamma(c_1 - c_2\gamma)$ leads to Verhulst Law [19], where $c_1, c_2 > 0$.
3. $\eta(\gamma) = -c\gamma^p$, where $c \geq 0$ and $0 < p < 1$ leads to Porous Media.

Note that the fractional derivative $\frac{\delta^\alpha}{\delta t^\alpha} = D_t^\alpha$ is described in the Caputo sense, which is defined in [1,2,20] as:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & \text{if } m-1 < \alpha \leq m, \\ f^{(m)}(t), & \text{if } \alpha = m \in \mathbb{N}, \end{cases} \tag{2.6}$$

where $f^{(m)}(t)$ denote $\frac{\partial^m f(t)}{\partial t^m}$ and Γ is the Gamma function defined as

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt, \tag{2.7}$$

which generalizes the factorial in the form:

$$\Gamma(m+1) = m! \tag{2.8}$$

The Caputo derivative has the following properties [1,2,20]:

1. $$D_t^\alpha c = 0,\ c \text{ is a constant},$$

2. $$D_t^\alpha t^m = \begin{cases} 0, & m \leq \alpha < 1, \\ \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}, & m > \alpha - 1. \end{cases}$$

3. $$\mathcal{L}_t\{D_t^\alpha f(t)\} = s^\alpha f(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0^+), \quad m-1 < \alpha \leq m,\ m \in \mathbb{N}. \tag{2.9}$$

The Laplace transform of Caputo fractional derivative requires the knowledge of the initial values of the function and its integer derivatives of order $k = 1, 2, \ldots, m-1$ [2], and when $\alpha \in (0,1]$ *i.e.* $0 < \alpha \leq 1$, (2.9) it is given by

$$\mathcal{L}_t\{D_t^\alpha f(t)\} = s^\alpha f(s) - f(0^+)s^{\alpha-1}. \tag{2.10}$$

3 The q-Laplace Homotopy Analysis Method

Given the fractional differential equation

$$D_t^\alpha \gamma(x,t) + \mathcal{B}(\gamma(x,t)) + \mathcal{N}(\gamma(x,t)) = \eta(x,t), \quad t \geq 0, \ n-1 < \alpha \leq n, \tag{3.1}$$

with the initial condition:

$$\gamma(x,0) = a, \tag{3.2}$$

where D_t^α is Caputo's derivative, $\mathcal{B}$ is a linear differential operator, $\mathcal{N}$ is a nonlinear differential operator, f is the source term and γ is the unknown function. By applying the Laplace transform in the variable t, denoted $\mathcal{L}_t$, to both sides of Eq.(3.1) we get

$$\mathcal{L}_t[\gamma(x,t)] - \frac{1}{s^\alpha}\sum_{k-1}^{i=0} s^{\alpha-i-1}\gamma^{(i)}(x,0) + \frac{1}{s^\alpha}\mathcal{L}_t[\mathcal{B}(\gamma(x,t)) + \mathcal{N}(\gamma(x,t)) - \eta(x,t)] = 0. \tag{3.3}$$

Using the initial condition (3.2), then we get

$$\mathcal{L}_t[\gamma(x,t)] - \frac{a}{s} + \frac{1}{s^\alpha}\mathcal{L}_t[\mathcal{B}(\gamma(x,t)) + \mathcal{N}(\gamma(x,t)) - \eta(x,t)] = 0. \tag{3.4}$$

In view of Liao's Homotopy Analysis Method [11, 12], for $0 \leq q \leq \frac{1}{n}, n \geq 1$, the zero-order deformation equation of the Laplace equation (3.4) has the form

$$(1-nq)\mathcal{L}_t(\tilde{\gamma}(x,t;q) - \gamma_0(x,t)) = \hbar q\mathcal{H}(x,t)\mathcal{J}[\gamma(x,t;q)], \tag{3.5}$$

where

$$\mathcal{J}[\gamma(x,t;q)] = \mathcal{L}_t[\gamma(x,t)] - \frac{a}{s} + \frac{1}{s^\alpha}\mathcal{L}_t[\mathcal{B}(\gamma(x,t)) + \mathcal{N}(\gamma(x,t)) - \eta(x,t)], \tag{3.6}$$

where $q \in [0, \frac{1}{n}]$ is the embedding parameter, $\mathcal{H} \neq 0$ is the nonzero auxiliary function, $\hbar \neq 0$ is an auxiliary parameter. Thus, from eq.(3.5), when $q = 0$ and $q = \frac{1}{n}$, we have

$$\tilde{\gamma}(x,t;0) = \gamma_0(x,t) \ \text{ and } \ \tilde{\gamma}\left(x,t;\frac{1}{n}\right) = \gamma(x,t) \tag{3.7}$$

respectively which explains the increment of q from 0 to $\frac{1}{n}$. That is, as q increases from 0 to $\frac{1}{n}$, the solution varies from the initial guess $\gamma_0(x,t)$ to the solution $\gamma(x,t)$. Expanding $\gamma(x,t;q)$ with respect to

q with the aid of Taylor's series expansion [13], we have

$$\tilde{\gamma}(x,t;q) = \gamma_0(x,t) + \sum_{j=1}^{\infty} \gamma_j(x,t) q^j, \tag{3.8}$$

where

$$\gamma_j(x,t) = \frac{1}{j!} \left. \frac{\partial^j \tilde{\gamma}(x,t;q)}{\partial q^j} \right|_{q=0}. \tag{3.9}$$

If $\gamma_0(x,t)$, $\hbar$ and $\mathcal{H}$ are properly chosen, then (3.8) converges at $q = \frac{1}{n}$ and we have

$$\tilde{\gamma}(x,t) = \gamma_0(x,t) + \sum_{m=1}^{\infty} \gamma_j(x,t) \left(\frac{1}{n}\right)^j. \tag{3.10}$$

Define the vector

$$\vec{\tilde{\gamma}}(x,t) = \{\gamma_r(x,t)\}_{r=0}^{j}. \tag{3.11}$$

By differentiating equation (3.5) m times with respect to q by using the Leibniz rule [18],

$$\begin{aligned} \sum_{n=0}^{m} \binom{m}{n} D^n (1-nq) D^{m-n} (\tilde{\gamma}(x,s;q) - \gamma_0(x,s)) = \\ \hbar\mathcal{H}(x,t) \sum_{n=0}^{m} \binom{m}{n} D^n q D^{m-n} \left[\gamma(x,s) - \frac{a}{s} + \frac{1}{s^\alpha} \mathcal{L}_t\{\mathcal{B}(\gamma(x,t)) - \mathcal{N}(\gamma(x,t)) - \eta(x,t)\} \right] \end{aligned} \tag{3.12}$$

setting $q = 0$, $\hbar = -1$, we have

$$\begin{aligned} D^m[\tilde{\gamma}(x,s;q) - \gamma_0(x,s)] - mD^{m-1}[\tilde{\gamma}(x,s;q) - \gamma_0(x,s)] = \\ -\mathcal{H}(x,t) m D^{m-1} \left[\gamma(x,s) - \frac{a}{s} + \frac{1}{s^\alpha} \mathcal{L}_t\{\mathcal{B}(\gamma(x,t)) - \mathcal{N}(\gamma(x,t)) - \eta(x,t)\} \right] \end{aligned} \tag{3.13}$$

and finally multiplying through by $\frac{1}{m!}$, we have the mth-order deformation equation given by

$$\gamma(x,s) - \chi_m^* \gamma_{m-1}(x,s) = -\mathcal{H}(x,t) \left[\gamma_{m-1}(x,s) - \left(1 - \frac{\chi_m^*}{n}\right) \left(\frac{a}{s} + \frac{1}{s^\alpha} \mathcal{L}_t \eta(x,t)\right) + \frac{1}{s^\alpha} \mathcal{L}_t\{\mathcal{B}(\gamma(x,t)) - H_{m-1}\} \right] \tag{3.14}$$

which can still be written as

$$\mathcal{L}_t[\gamma_m(x,t) - \chi_m^* \gamma_{m-1}(x,t)] = -\mathcal{H}\mathcal{R}_m(\vec{\gamma}_{m-1}(x,t)), \tag{3.15}$$

where

$$\mathcal{R}_m(\vec{\gamma}_{m-1}(x,t)) = \mathcal{L}_t[\gamma_{m-1}(x,t)] - \left(\frac{a}{s} + \frac{1}{s^\alpha} \mathcal{L}_t[\eta(x,t)]\right) \left(1 - \frac{\chi_m^*}{n}\right) + \frac{1}{s^\alpha} \mathcal{L}_t[B(\gamma(x,t)) + H_{m-1}] \tag{3.16}$$

with H denoting the homotopy polynomial defined as

$$H_m = \frac{1}{m!}\left.\frac{\partial^m \tilde{\gamma}(x,t;q)}{\partial q^m}\right|_{q=0}, \quad \tilde{\gamma}(x,t;q) = \tilde{\gamma}_0 + q\gamma_1 + q^2\tilde{\gamma}_2 + \dots \tag{3.17}$$

and

$$\chi_m^* = \begin{cases} 0, & \text{if } m \leq 1 \\ n, & \text{if } m > 1. \end{cases}$$

Taking the inverse Laplace transform, that is $\mathcal{L}_t^{-1}$, of (3.15) we have

$$\gamma_m(x,t) = \chi_m^*\gamma_{m-1}(x,t) - \mathcal{L}_t^{-1}[\mathcal{H}(x,t)\mathcal{R}_m(\vec{\gamma}_{m-1}(x,t))]. \tag{3.18}$$

We have outlined the step-by-step procedure for qLHAM above and it is obeservable that when $n = 1$ from equations (3.5) to (3.18) the procedure reduces to the method described in [17, 18]. Suppose there exists a constant κ, $0 < \kappa < 1$ such that

$$||\gamma_{m+1}(x,t)|| \leq \kappa||\gamma_m(x,t)|| \tag{3.19}$$

for each value of m. If we truncate the series solution in eq. (3.18), the truncated series

$$\sum_{m=0}^{i} \gamma_m(x,t)\left(\frac{1}{n}\right)^m$$

is an approximate solution of $\gamma(x,t)$ and the maximum absolute error can be derived by using

$$\left|\left|\gamma(x,t) - \sum_{m=0}^{i}\gamma_m(x,t)\left(\frac{1}{n}\right)^m\right|\right| \leq \frac{\kappa^{i+1}}{n^i(n-\kappa)}||\gamma_0(x,y,t)||. \tag{3.20}$$

For detailed proof of the convergence analysis and error analysis, interested reader can refer to [14–16] and some of their cited references.

4 Implementation

In this section, qLHAM is applied to determine the exact solution of some special cases of (2.5).

Example 4.1. [3–10, 15] Consider the following generalised Caputo time-fractional biological population model

$$D_t^\alpha\gamma(x,y,t) - \gamma_{xx}^2(x,y,t) - \gamma_{yy}^2(x,y,t) = \eta\gamma(x,y,t), \; 0 < \alpha \leq 1, t > 0, \tag{4.1}$$

with the initial condition

$$\gamma(x,y,0) = \sqrt{xy}. \tag{4.2}$$

Applying Laplace transform to both sides of (4.1) using (4.2), we have:

$$\gamma(x,y,s) - \frac{\sqrt{xy}}{s} - \frac{1}{s^\alpha}\mathcal{L}_t[\gamma_{xx}^2(x,y,t) + \gamma_{yy}^2(x,y,t) + \eta\gamma(x,y,t)] = 0, \tag{4.3}$$

differentiating (4.3) m-times with respect to q

$$\sum_{n=0}^{m}\binom{m}{n} D^n(1-nq)D^{m-n}(\tilde{\gamma}(x,y,s;q) - \gamma_0(x,y,s)) = \hbar\mathcal{H}(x,y,t)\sum_{n=0}^{m}\binom{m}{n} D^n q D^{m-n}\left[\tilde{\gamma}(x,y,s;q) - \frac{\sqrt{xy}}{s} - \frac{1}{s^\alpha}\mathcal{L}_t\{\gamma_{xx}^2(x,y,t) + \gamma_{yy}^2(x,y,t) + \eta\gamma(x,y,t)\}\right] \tag{4.4}$$

setting $q = 0$, $\hbar = -1$ and $\mathcal{H}(x,y,t) = 1$,

$$D^m[\tilde{\gamma}(x,y,s;q) - \gamma_0(x,y,s)] - mD^{m-1}[\tilde{\gamma}(x,y,s;q) - \gamma_0(x,y,s)] = -mD^{m-1}\left[\tilde{\gamma}(x,y,s;q) - \frac{\sqrt{xy}}{s} - \frac{1}{s^\alpha}\mathcal{L}_t\{\gamma_{xx}^2(x,y,t) + \gamma_{yy}^2(x,y,t) + \eta\gamma(x,y,t)\}\right] \tag{4.5}$$

multiply through by $\frac{1}{m!}$

$$\frac{1}{m!}D^m\,\tilde{\gamma}(x,y,s;q)|_{q=0} - \frac{1}{(m-1)!}D^{m-1}\,\tilde{\gamma}(x,y,s;q)|_{q=0} = -\left[\frac{1}{(m-1)!}\tilde{\gamma}(x,y,s;q) - \frac{1}{(m-1)!}D^{m-1}\left(\frac{\sqrt{xy}}{s} + \frac{\eta}{s^\alpha}\mathcal{L}_t(\gamma(x,y,t))\right) - \frac{1}{s^\alpha}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\right] \tag{4.6}$$

which implies

$$\gamma_m(x,y,s) - \chi_m^*\gamma_{m-1}(x,y,s) = -\left[\gamma_{m-1}(x,y,s) - \left(\frac{\sqrt{xy}}{s} + \frac{\eta}{s^\alpha}\mathcal{L}_t(\gamma_{m-1}(x,y,t))\right)\left(1 - \frac{\chi_m^*}{n}\right) - \frac{1}{s^\alpha}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\right] \tag{4.7}$$

therefore,

$$\gamma_m(x,y,s) = \chi_m^*\gamma_{m-1}(x,y,s) - \left[\gamma_{m-1}(x,y,s) - \left(\frac{\sqrt{xy}}{s} + \frac{\eta}{s^\alpha}\mathcal{L}_t(\gamma_{m-1}(x,y,t))\right)\left(1 - \frac{\chi_m^*}{n}\right) - \frac{1}{s^\alpha}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\right]. \tag{4.8}$$

Inverse Laplace transform of (4.8) is the equation:

$$\begin{aligned}\gamma_m(x,y,t) = \\ &\chi_m^*\gamma_{m-1}(x,y,t) - \mathcal{L}_t^{-1}\Bigg[\gamma_{m-1}(x,y,s) - \left(\frac{\sqrt{xy}}{s} + \frac{\eta}{s^\alpha}\mathcal{L}_t(\gamma_{m-1}(x,y,t))\right)\left(1-\frac{\chi_m^*}{n}\right) \\ &- \frac{1}{s^\alpha}\mathcal{L}_t^{-1}\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\Bigg] \qquad (4.9)\end{aligned}$$

where

$$\chi_m^* = \begin{cases} 0, & \text{if } m \leq 1 \\ n, & \text{if } m > 1 \end{cases}. \qquad (4.10)$$

Using eq. (4.9) and condition (4.10), we have the following iterations:

$$\begin{aligned}\gamma_0(x,y,t) &= \sqrt{xy} = \gamma_0, \\ \gamma_1(x,y,t) &= \frac{\eta t^\alpha\sqrt{xy}}{\Gamma(\alpha+1)} = \gamma_0\frac{\eta t^\alpha}{\Gamma(\alpha+1)}, \\ \gamma_2(x,y,t) &= \frac{\eta^2 t^{2\alpha}\sqrt{xy}}{\Gamma(2\alpha+1)} = \gamma_0\frac{\eta^2 t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ \gamma_3(x,y,t) &= \frac{\eta^3 t^{3\alpha}\sqrt{xy}}{\Gamma(3\alpha+1)} = \gamma_0\frac{\eta^3 t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &\vdots\end{aligned}$$

which implies that,

$$\begin{aligned}\gamma(x,y,t) = \sum_{m=0}^{\infty}\gamma_m(x,y,t) &= \sqrt{xy} + \frac{\eta t^\alpha\sqrt{xy}}{\Gamma(\alpha+1)} + \frac{\eta^2 t^{2\alpha}\sqrt{xy}}{\Gamma(2\alpha+1)} + \frac{\eta^3 t^{3\alpha}\sqrt{xy}}{\Gamma(3\alpha+1)} + \dots \\ &= \sqrt{xy}\left(1 + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\eta^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\eta^3 t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right) \qquad (4.11) \\ &= \sqrt{xy}E_{\alpha,1}(\eta t^\alpha).\end{aligned}$$

If we set $\alpha = 1$ in eq. (4.11), we have:

$$\gamma(x,y,t) = \sqrt{xy}\left(1 + \frac{\eta t}{\Gamma 2} + \frac{\eta^2 t^2}{\Gamma 3} + \frac{\eta^3 t^3}{\Gamma 4} + \dots\right). \qquad (4.12)$$

Using (2.8) and (2.4), (4.12) reduces to

$$\gamma(x,y,t) = \sqrt{xy}\left(1 + \eta t + \frac{\eta^2 t^2}{2!} + \frac{\eta^3 t^3}{3!} + \dots\right) \equiv \sqrt{xy}e^{\eta t}. \qquad (4.13)$$

Example 4.2. [4, 6] Consider the following generalised Caputo time-fractional biological population model

$$D_t^{\alpha}\gamma(x,y,t) - \gamma_{xx}^2(x,y,t) - \gamma_{yy}^2(x,y,t) = \eta\gamma^{-1}(1-a\gamma),\ 0<\alpha\leq 1, t>0, \tag{4.14}$$

subject to the initial condition

$$\gamma(x,y,t) = \sqrt{\frac{\eta a}{4}x^2 + \frac{\eta a}{4}y^2 + y + 5}. \tag{4.15}$$

Laplace transform of both sides of (4.14) using (4.15) as expressed in (2.10) is given by:

$$\gamma(x,y,s) - \frac{1}{s}\sqrt{\frac{\eta a}{4}x^2 + \frac{\eta a}{4}y^2 + y + 5} - \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\gamma_{xx}^2(x,y,t) + \gamma_{xx}^2(x,y,t) + \eta\gamma^{-1}(1-a\gamma)\right] = 0. \tag{4.16}$$

The nonlinear operator is thus

$$\mathcal{N}[\tilde{\gamma}(x,y,t;q)] = \tilde{\gamma}(x,y,s;q) - \frac{1}{s}\sqrt{\frac{\eta a}{4}x^2 + \frac{\eta a}{4}y^2 + y + 5} - \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\tilde{\gamma}_{xx}^2(x,y,t) + \tilde{\gamma}_{xx}^2(x,y,t) + \eta\tilde{\gamma}^{-1}(1-a\tilde{\gamma})(x,y,t;q)\right]. \tag{4.17}$$

For the m-th order deformation equation, we differentiate (4.17) m times wrt q. In the process, we set $q=0$, $\hbar=-1$ and $\mathcal{H}(x,y,t)=1$. Finally multiplying by $\frac{1}{m!}$, we obtain:

$$\begin{aligned}\gamma_m(x,y,s) - \chi_m^*\gamma_{m-1}(x,y,s) = \\ -\Bigg[\gamma_{m-1}(x,y,s) - \left(\frac{1}{s}\sqrt{\frac{\eta a}{4}x^2 + \frac{\eta a}{4}y^2 + y + 5} + \frac{\eta}{s^{\alpha}}\mathcal{L}_t(\gamma_{m-1}^{-1}(1-a\gamma_{m-1})(x,y,t))\right)\left(1-\frac{\chi_m^*}{n}\right) \\ - \frac{1}{s^{\alpha}}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\Bigg].\end{aligned} \tag{4.18}$$

Inverse Laplace transform of (4.18) gives the equation

$$\begin{aligned}\gamma_m(x,y,t) - \chi_m^*\gamma_{m-1}(x,y,t) = \\ -\Bigg[\gamma_{m-1}(x,y,t) - \mathcal{L}_t^{-1}\left\{\left(\frac{1}{s}\sqrt{\frac{\eta a}{4}x^2 + \frac{\eta a}{4}y^2 + y + 5} + \frac{\eta}{s^{\alpha}}\mathcal{L}_t(\gamma_{m-1}^{-1}(1-a\gamma_{m-1})(x,y,t))\right)\left(1-\frac{\chi_m^*}{n}\right)\right\} \\ - \mathcal{L}_t^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\right\}\Bigg]\end{aligned} \tag{4.19}$$

$$\chi_m^* = \begin{cases} 0, & \text{if } m \leq 1 \\ n, & \text{if } m > 1. \end{cases}. \tag{4.20}$$

Using (4.19) and (4.20) we get the following iterations for $m = 0, 1, 2, 3, \dots$

$$\gamma_0(x,y,t) = \sqrt{\frac{\eta a}{4}x^2 + \frac{\eta a}{4}y^2 + y + 5} = \gamma_0,$$

$$\gamma_1(x,y,t) = \frac{\eta t^\alpha}{\Gamma(\alpha+1)\gamma_0},$$

$$\gamma_2(x,y,t) = \frac{-2\eta^2 t^{2\alpha}}{\Gamma(2\alpha+1)\gamma_0^3},$$

$$\gamma_3(x,y,t) = \frac{3\eta^3 t^{3\alpha}}{\Gamma(3\alpha+1)\gamma_0^5},$$

$$\vdots$$

which implies that

$$\gamma(x,y,t) = \sum_{m=0}^{\infty} \gamma_m(x,y,t) = \gamma_0 + \frac{\eta t^\alpha}{\gamma_0}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(2\alpha+1)}\left(\frac{-\eta t^\alpha}{\gamma_0^2}\right) + \frac{3}{\Gamma(3\alpha+1)}\left(\frac{\eta^2 t^{2\alpha}}{\gamma_0^4}\right) + \dots\right] \tag{4.21}$$

and can be rewritten as

$$\gamma(x,y,t) = \gamma_0 + \frac{\eta t^\alpha}{\gamma_0}\sum_{m=1}^{\infty} \frac{m}{\Gamma(m\alpha+1)}\left(\frac{-\eta t^\alpha}{\gamma_0^2}\right)^{m-1}. \tag{4.22}$$

When $\alpha = 1$, (4.22) becomes

$$\begin{aligned} \gamma(x,y,t) &= \gamma_0 + \frac{\eta t}{\gamma_0}\left[\frac{1}{\Gamma 2} + \frac{-\eta t}{\gamma_0^2}\frac{2}{\Gamma 3} + \frac{\eta^2 t^2}{\eta_0^4}\frac{3}{\Gamma 4} + \dots\right], \\ &= \gamma_0 + \frac{\eta t}{\gamma_0}\left[1 + \frac{-\eta t}{\gamma_0^2} + \frac{\eta^2 t^2}{\eta_0^4}\frac{1}{2!} + \dots\right], \\ &= \gamma_0 + \frac{\eta t}{\gamma_0}E_{1,1}\left(\frac{-\eta t}{\gamma_0^2}\right) = \gamma_0 + \frac{\eta t}{\gamma_0}exp\left(\frac{-\eta t}{\gamma_0^2}\right). \end{aligned} \tag{4.23}$$

Example 4.3. [3–10] Consider the following generalised Caputo time-fractional biological population model

$$D_t^\alpha \gamma(x,y,t) - \gamma_{xx}^2(x,y,t) - \gamma_{yy}^2(x,y,t) = \gamma(x,y,t),\ 0 < \alpha \le 1, t > 0, \tag{4.24}$$

subject to the initial condition

$$\gamma(x,y,0) = \sqrt{\sin x \sinh y}. \tag{4.25}$$

The Laplace transform of both sides of (4.24) and (4.25) as defined in (2.10) is the equation

$$\gamma(x,y,s) - \frac{1}{s}\sqrt{\sin x \sinh y} - \frac{1}{s^\alpha}\mathcal{L}_t\left[\gamma_{xx}^2(x,y,t) + \gamma_{xx}^2(x,y,t) + \gamma(x,y,t)\right] = 0 \tag{4.26}$$

with the m-th order deformation equation

$$\gamma_m(x,y,s) - \chi_m^* \gamma_{m-1}(x,y,s) = \\ - \left[\gamma_{m-1}(x,y,s) - \left(\frac{1}{s}\sqrt{\sin x \sinh y} + \frac{1}{s^\alpha}\mathcal{L}_t(\gamma_{m-1}(x,y,t)) \right) \left(1 - \frac{\chi_m^*}{n}\right) \right. \\ \left. - \frac{1}{s^\alpha}\mathcal{L}_t\left(\sum_{i=0}^{m-1} (\gamma_i \gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1} (\gamma_i \gamma_{m-1-i})_{yy}(x,y,t) \right) \right] \tag{4.27}$$

where already we have set $q = 0$, $\hbar = -1$ and $\mathcal{H}(x,y,t) = 1$. Inverse Laplace transformof (4.27) is the equation:

$$\gamma_m(x,y,t) - \chi_m^* \gamma_{m-1}(x,y,t) = \\ - \left[\gamma_{m-1}(x,y,t) - \mathcal{L}_t^{-1}\left\{ \left(\frac{1}{s}\sqrt{\sin x \sinh y} + \frac{1}{s^\alpha}\mathcal{L}_t(\gamma_{m-1}(x,y,t)) \right) \left(1 - \frac{\chi_m^*}{n}\right) \right\} \right. \\ \left. - \mathcal{L}_t^{-1}\left\{ \frac{1}{s^\alpha}\mathcal{L}_t\left(\sum_{i=0}^{m-1} (\gamma_i \gamma_{m-1-i})_{xx}(x,y,t) + \sum_{i=0}^{m-1} (\gamma_i \gamma_{m-1-i})_{yy}(x,y,t) \right) \right\} \right] \tag{4.28}$$

$$\chi_m^* = \begin{cases} 0, & \text{if } m \leq 1 \\ n, & \text{if } m > 1. \end{cases} \tag{4.29}$$

Using (4.28) and (4.29) for the first few terms of our solution, we obtain:

$$\gamma_0(x,y,t) = \sqrt{\sin x \sinh y} = \gamma_0,$$

$$\gamma_1(x,y,t) = \sqrt{\sin x \sinh y}\frac{t^\alpha}{\Gamma(\alpha+1)} = \gamma_0 \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$\gamma_2(x,y,t) = \sqrt{\sin x \sinh y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = \gamma_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\gamma_3(x,y,t) = \sqrt{\sin x \sinh y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = \gamma_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

$$\gamma(x,y,t) = \sum_{m=0}^{\infty} \gamma_m(x,y,t) = \gamma_0 \left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right]$$

which can also be written as

$$\gamma(x,y,t) = \gamma_0 \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{4.30}$$

If, for a test case, we set $\alpha = 1$ in (4.30), we obtain:

$$\gamma(x,y,t) = \gamma_0 \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)}. \tag{4.31}$$

Example 4.4. [3,4,6–10] Consider the following generalised Caputo time-fractional biological population model

$$D_t^{\alpha}\gamma(x,y,t)-\gamma_{xx}^2(x,y,t)-\gamma_{yy}^2(x,y,t)=\gamma(1-a\gamma),\ 0<\alpha\leq 1, t>0, \tag{4.32}$$

subject to the initial condition

$$\gamma(x,y,0)=exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]. \tag{4.33}$$

Applying Laplace transform to both sides of (4.32) using (4.33), we have

$$\gamma(x,y,s)-\frac{1}{s}exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]-\frac{1}{s^{\alpha}}\mathcal{L}_t\left[\gamma_{xx}^2(x,y,t)+\gamma_{xx}^2(x,y,t)+\gamma(1-a\gamma)(x,y,t)\right]=0 \tag{4.34}$$

with m-th order deformation equation

$$\begin{aligned}&\gamma_m(x,y,s)-\chi_m^*\gamma_{m-1}(x,y,s)=\\&-\left[\gamma_{m-1}(x,y,s)-\left(\frac{1}{s}exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]+\frac{1}{s^{\alpha}}\mathcal{L}_t(\gamma_{m-1}(1-a\gamma_{m-1})(x,y,t))\right)\left(1-\frac{\chi_m^*}{n}\right)\right.\\&\left.-\frac{1}{s^{\alpha}}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t)+\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\right]\end{aligned} \tag{4.35}$$

with $q,\hbar$ and $\mathcal{H}(x,y,t)$ set to $0,-1$ and 1 respectively. Inverse Laplace transform of (4.35) is thus given by

$$\begin{aligned}&\gamma_m(x,y,t)-\chi_m^*\gamma_{m-1}(x,y,t)=\\&-\left[\gamma_{m-1}(x,y,t)-\mathcal{L}_t^{-1}\left\{\left(\frac{1}{s}exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]+\frac{1}{s^{\alpha}}\mathcal{L}_t(\gamma_{m-1}(1-a\gamma_{m-1})(x,y,t))\right)\left(1-\frac{\chi_m^*}{n}\right)\right\}\right.\\&\left.-\mathcal{L}_t^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}_t\left(\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{xx}(x,y,t)+\sum_{i=0}^{m-1}(\gamma_i\gamma_{m-1-i})_{yy}(x,y,t)\right)\right\}\right]\end{aligned} \tag{4.36}$$

$$\chi_m^*=\begin{cases}0, & \text{if } m\leq 1\\ n, & \text{if } m>1.\end{cases} \tag{4.37}$$

Using (4.36) and (4.37) for the first few terms of our solution, we obtain:

$$\gamma_0(x,y,t)=exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]=\gamma_0,$$

$$\gamma_1(x,y,t)=exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}=\gamma_0\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$\gamma_2(x,y,t)=exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}=\gamma_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\gamma_3(x,y,t) = exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = \gamma_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

$$\gamma(x,y,t) = \sum_{m=0}^{\infty}\gamma_m(x,y,t) = \gamma_0\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots\right]$$

$$\begin{aligned}\gamma(x,y,t) = \sum_{m=0}^{\infty}\gamma_m(x,y,t) &= \gamma_0\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots\right]\\ &= \gamma_0\sum_{m=0}^{\infty}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}.\end{aligned} \tag{4.38}$$

For $\alpha = 1$, (4.38) becomes

$$\gamma(x,y,t) = \gamma_0\sum_{m=0}^{\infty}\frac{t^{m}}{\Gamma(m+1)}. \tag{4.39}$$

(4.39) can be reduced to the exact solution

$$\begin{aligned}\gamma(x,y,t) &= \gamma_0 E_{1,1}(t) = \gamma_0 e^{t}\\ &= exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)\right]e^{t}\\ &= exp\left[\frac{1}{2}\sqrt{\frac{a}{2}}(x+y)+t\right],\end{aligned} \tag{4.40}$$

where $E_{1,1}(t)$ is the Mittag-Leffler function, a prominent generalization of the exponential function. It is generally defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\alpha k+\beta)},$$

where $\alpha > 0$ and $\beta > 0$. In some of our examples, we have $\alpha = \beta = 1$, which reduces to the exponential function:

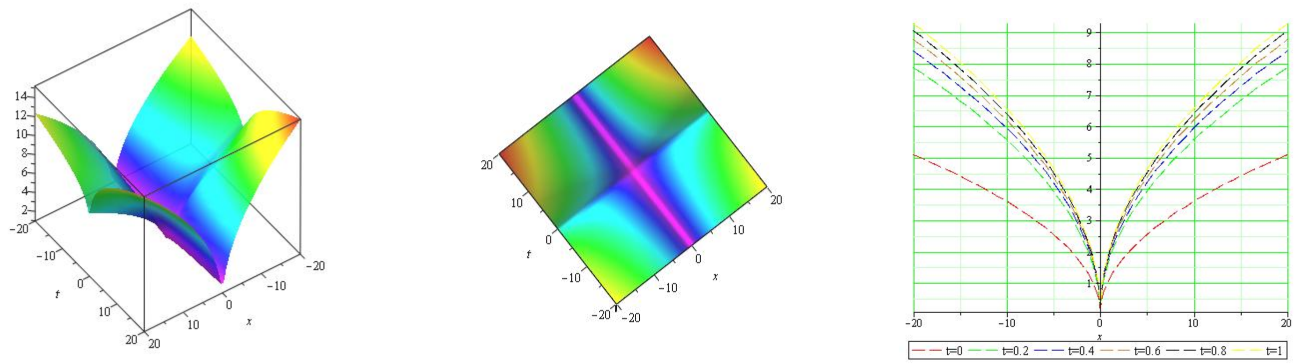
$$E_{1,1}(t) = \sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{t^{k}}{k!} = e^{t}.$$

The Mittag-Leffler function appears in fractional calculus to model processes with memory and hereditary properties – a major advantage of fractional calculus and its use [1,2].

5 Discussion of Results

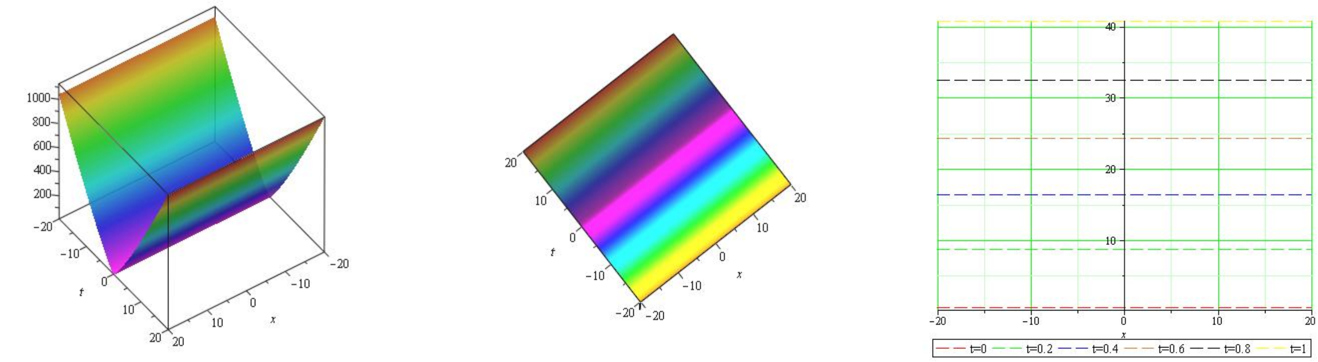
As shown in the examples above, we have added to the list of methods applied to this particular biological population model in the fractional order sense and by extension the integer order when $\alpha = 1$. Compared to other methods cited in the examples we are not concerned with other values of $\hbar$ thus have only solved for the value of $\hbar = -1$. Other methods from [3–10, 15] outlined a procedure for the values of $\hbar$ which can be used to plot the $\hbar$-curve – see [15] for example. A very close method to the method we used in this paper is the one described in the 2013 paper by [8] where they applied Homotopy Analysis Transform Method (HATM), a combination of Laplace transformation and the Homotopy Analysis Method described in [11, 12]. What distincts our work from theirs is our method applied a different approach allowed by the value of q which changes the course of solution and its rate of convergence. For HATM, $q \in (0, 1]$ depending on the condition of the model while $q \in (0, \frac{1}{n}]$ for qLHAM. Example 4.1 had already been solved in [15] but we also show it here so that we compare the solution with those of others order than the ones described in [15].

We displayed the graphs of all the problems with different fractional orders in this section. Graphing the 2D, 3D, and contour plots of the provided problems using appropriate values. The dynamical structures of the fractional biological population model were explained by these graphs:



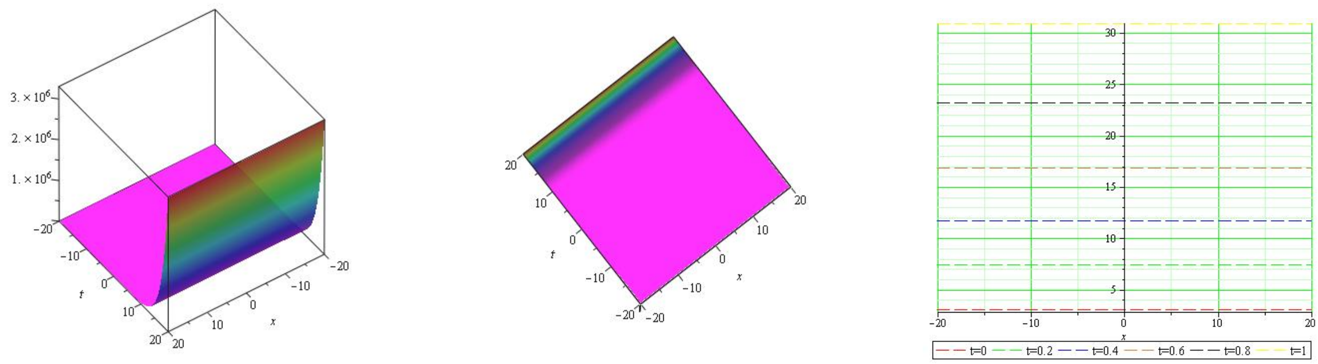
(a) 3D-plot (b) Contour plot (c) 2D-plot

Figure 5.1: The absolute graph of Example 4.1 with $\eta = 0.6$, $\alpha = 0.2$ and $y = -1.3$.



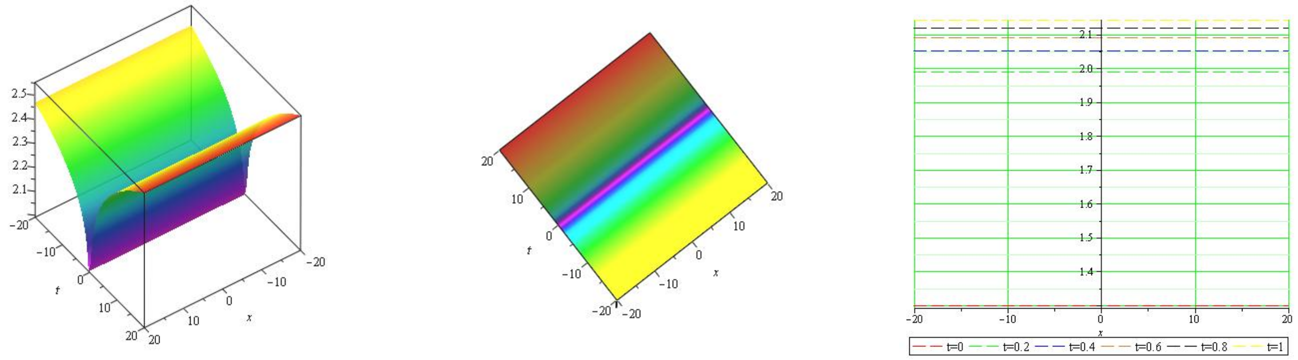
(a) $3D$-plot (b) Contour plot (c) $2D$-plot

Figure 5.2: The absolute graph of Example 4.2 with $\eta = 1.3$, $\alpha = 0.4$ and $\gamma_0 = 0.6$.



(a) $3D$-plot (b) Contour plot (c) $2D$-plot

Figure 5.3: The absolute graph of Example 4.3 with $\eta = 2.3$, $\alpha = 0.6$ and $\gamma_0 = 3.1$.



(a) 3D-plot (b) Contour plot (c) 2D-plot

Figure 5.4: The absolute graph of Example 4.4 with $\eta = 0.5$, $\alpha = 0.1$ and $\gamma_0 = -1.3$.

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