Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 2, Number 1, 2019, Pages 87-99
https://doi.org/10.34198/ejms.2119.8799

A Certain Subclass of Analytic and Univalent Functions Defined by Hadamard Product

Dhirgam Allawy Hussein[1] **and Sahar Jaafar Mahmood**[2]

[1]Directorate of Education in Al-Qadisiyah, Diwaniyah, Iraq; e-mail: dhirgam82@gmail.com

[2]Department of Mathematics, College of Computer Science and Information Technology, University of Al-Qadisiyah, Iraq; e-mail: sahar.abumalah@qu.edu.iq

Abstract

In this paper, we present a new subclass $AD(\lambda, \gamma, \alpha, \beta)$ of analytic univalent functions in the open unit disk U. We establish some interesting properties like, coefficient estimates, closure theorems, extreme points, growth and distortion theorem and radius of starlikeness and convexity.

1. Introduction and Preliminaries

Let A denote the class of analytic functions f defined on unit disk $\boldsymbol{U} = \{z \in \boldsymbol{C} : |z| = \mathbf{1}\}$ with normalization. Such a function has the form $f(0) = 0$, $f'(0) = 1$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in U \tag{1.1}$$

and the convolution (Hadamard Product) $(f * g)$ of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(x),$$

Received: March 3, 2019; Revised: April 26, 2019; Accepted: April 29, 2019

2010 Mathematics Subject Classification: 30C45.

Keywords and phrases: analytic functions, univalent function, Hadamard product.

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \ z \in U.$

Definition 1.1. A function $f \in A$ is in the class $AD(\lambda, \gamma, \alpha, \beta)$ if it satisfies the analytic criteria

$$\left| \frac{\lambda\gamma z^3 (f * g)'''(z) + (\lambda + \gamma(2\lambda - 1)) z^2 (f * g)''(z) + z(f * g)'(z)}{\lambda\gamma z^2 (f * g)''(z) + (\lambda - \gamma) z (f * g)'(z) + (1 - \lambda + \gamma)(f * g)(z)} \right| < \frac{\alpha\beta}{2}, \quad (1.2)$$

where $0 \le \gamma \le \lambda \le 1, \ 0 \le \beta \le 1, \ \alpha \in C/\{0\}, \ z \in U.$

Many authors were studied another classes defined in U, like, Darus [1], Goodman [2], Rosy [3] and, Sunil Varma and Rosy [4].

We study many interesting properties on our class as follows:

2. Coefficient Estimates

We now prove the coefficient estimates for function in the class $AD(\lambda, \gamma, \alpha, \beta)$.

Theorem 2.1. *A function f of the form* (1.1) *is in the class* $AD(\lambda, \gamma, \alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} [(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n - 1) + 1)] a_n b_n$$

$$\le \alpha\beta - 2, \quad (2.1)$$

where $0 \le \gamma \le \lambda \le 1, \ 0 \le \beta \le 1, \ \alpha \in C/\{0\}, \ z \in U.$

Proof. Assume that inequality (2.1) holds true and $|z| = 1$, then

$$2|\lambda\gamma z^3 (f * g)'''(z) + (\lambda + \gamma(2\lambda - 1)) z^2 (f * g)''(z) + z(f * g)'(z)|$$

$$- \alpha\beta|\lambda\gamma z^2 (f * g)''(z) + (\lambda - \gamma) z (f * g)'(z) + (1 - \lambda + \gamma)(f * g)(z)|$$

$$= 2\left| \lambda\gamma \sum_{n=2}^{\infty} n(n-1)(n-2) a_n b_n z^n + (\lambda + \gamma(2\lambda - 1)) \sum_{n=2}^{\infty} n(n-1) a_n b_n z^n + z + \sum_{n=2}^{\infty} n a_n b_n z^n \right|$$

$$- \alpha\beta \left| \lambda\gamma \sum_{n=2}^{\infty} n(n-1) a_n b_n z^n + z(\lambda - \gamma) + (\lambda - \gamma) \sum_{n=2}^{\infty} n a_n b_n z^n + z(1 - \lambda + \gamma) \right.$$

$$\left. + (1 - \lambda + \gamma) \sum_{n=2}^{\infty} a_n b_n z^n \right|$$

$$\leq 2\lambda\gamma \sum_{n=2}^{\infty} n(n-1)(n-2) a_n b_n + 2(\lambda + \gamma(2\lambda - 1)) \sum_{n=2}^{\infty} n(n-1) a_n b_n + 2$$

$$+ 2 \sum_{n=2}^{\infty} n a_n b_n - \alpha\beta\lambda\gamma \sum_{n=2}^{\infty} n(n-1) a_n b_n$$

$$- \alpha\beta(\lambda - \gamma) - \alpha\beta(\lambda - \gamma) \sum_{n=2}^{\infty} n a_n b_n - \alpha\beta(1 - \lambda + \gamma) - \alpha\beta(1 - \lambda + \gamma) \sum_{n=2}^{\infty} a_n b_n$$

$$= \sum_{n=2}^{\infty} \begin{bmatrix} 2\lambda\gamma n(n-1)(n-2) + 2(\lambda + \gamma(2\lambda - 1)) n(n-1) + 2n \\ - \alpha\beta\lambda\gamma n(n-1) - \alpha\beta(\lambda - \gamma) n - \alpha\beta(1 - \lambda + \gamma) \end{bmatrix} a_n b_n$$

$$+ 2 - \alpha\beta(\lambda - \gamma) - \alpha\beta(1 - \lambda + \gamma)$$

$$= \sum_{n=2}^{\infty} [(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1) + 1)] a_n b_n + 2 - \alpha\beta$$

$$\leq 0$$

by hypothesis and by maximum modulus principle, then $f \in AD(\lambda, \gamma, \alpha, \beta)$.

Conversely: Let $f \in AD(\lambda, \gamma, \alpha, \beta)$. Then

$$\left| \frac{\lambda\gamma z^3 (f * g)'''(z) + (\lambda + \gamma(2\lambda - 1)) z^2 (f * g)''(z) + z(f * g)'(z)}{\lambda\gamma z^2 (f * g)''(z) + (\lambda - \gamma) z (f * g)'(z) + (1 - \lambda + \gamma)(f * g)(z)} \right| < \frac{\alpha\beta}{2}.$$

That is,

$$\left|\frac{\lambda\gamma\sum_{n=2}^{\infty}n(n-1)(n-2)a_nb_nz^n+(\lambda+\gamma(2\lambda-1))\sum_{n=2}^{\infty}n(n-1)a_nb_nz^n+z+\sum_{n=2}^{\infty}na_nb_nz^n}{\lambda\gamma\sum_{n=2}^{\infty}n(n-1)a_nb_nz^n+z(\lambda-\gamma)+(\lambda-\gamma)\sum_{n=2}^{\infty}na_nb_nz^n+z(1-\lambda+\gamma)+(1-\lambda+\gamma)\sum_{n=2}^{\infty}a_nb_nz^n}\right|<\frac{\alpha\beta}{2}.$$

Since $\mathrm{Re}(z)\le|z|$ for all z, we have

$$\mathrm{Re}\left[\frac{\lambda\gamma\sum_{n=2}^{\infty}n(n-1)(n-2)a_nb_nz^n+(\lambda+\gamma(2\lambda-1))\sum_{n=2}^{\infty}n(n-1)a_nb_nz^n+z+\sum_{n=2}^{\infty}na_nb_nz^n}{\lambda\gamma\sum_{n=2}^{\infty}n(n-1)a_nb_nz^n+z(\lambda-\gamma)+(\lambda-\gamma)\sum_{n=2}^{\infty}na_nb_nz^n+z(1-\lambda+\gamma)+(1-\lambda+\gamma)\sum_{n=2}^{\infty}a_nb_nz^n}\right]\le\frac{\alpha\beta}{2}.$$

Choosing z on real axis and allowing $z\to 1^-$, we have

$$\frac{\lambda\gamma\sum_{n=2}^{\infty}n(n-1)(n-2)a_nb_n+(\lambda+\gamma(2\lambda-1))\sum_{n=2}^{\infty}n(n-1)a_nb_n+1+\sum_{n=2}^{\infty}na_nb_n}{\lambda\gamma\sum_{n=2}^{\infty}n(n-1)a_nb_n+(\lambda-\gamma)+(\lambda-\gamma)\sum_{n=2}^{\infty}na_nb_n+(1-\lambda+\gamma)+(1-\lambda+\gamma)\sum_{n=2}^{\infty}a_nb_n}$$

$$\le\frac{\alpha\beta}{2}.$$

That is,

$$2\lambda\gamma\sum_{n=2}^{\infty} n(n-1)(n-2)a_n b_n + 2(\lambda+\gamma(2\lambda-1))\sum_{n=2}^{\infty} n(n-1)a_n b_n + 2 + 2\sum_{n=2}^{\infty} na_n b_n$$

$$\leq -\alpha\beta\lambda\gamma\sum_{n=2}^{\infty} n(n-1)a_n b_n - \alpha\beta(\lambda-\gamma) - \alpha\beta(\lambda-\gamma)\sum_{n=2}^{\infty} na_n b_n - \alpha\beta(1-\lambda+\gamma)$$

$$-\alpha\beta(1-\lambda+\gamma)\sum_{n=2}^{\infty} a_n b_n$$

$$= \sum_{n=2}^{\infty}[(n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma)) + 2n - \alpha\beta((\lambda-\gamma)(n-1)+1)]a_n b_n$$

$$\leq \alpha\beta - 2$$

which obviously is required assertion (2.1).

Finally, sharpness follows if we take

$$f(z) = z + \frac{\alpha\beta-2}{((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma))+2n-\alpha\beta((\lambda-\gamma)(n-1)+1))b_n} z^n. \quad (2.2)$$

Corollary 2.1. *If* $f \in AD(\lambda, \gamma, \alpha, \beta)$, *then*

$$a_n \leq \frac{\alpha\beta-2}{((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma))+2n-\alpha\beta((\lambda-\gamma)(n-1)+1))b_n},$$

$$n = 2, 3, \ldots . \quad (2.3)$$

The equality in (2.3) is attained for the function $f(z)$ given by (2.2).

3. Closure Theorems

Theorem 3.1. *The class* $AD(\lambda, \gamma, \alpha, \beta)$ *is convex.*

Proof. Let f_1, f_2 be two functions in $AD(\lambda, \gamma, \alpha, \beta)$. Then

$$f_1(z) = z + \sum_{n=2}^{\infty} a_{n,1} z^n$$

$$f_2(z) = z + \sum_{n=2}^{\infty} a_{n,2} z^n.$$

Define $g(z) = Cf_1(z) + (1 - C) f_2(z),\ 0 \le C \le 1,$ then

$$g(z) = z + \sum_{n=2}^{\infty} [Ca_{n,1} + (1 - C) a_{n,2}] z^n.$$

Now:

$$\sum_{n=2}^{\infty} [(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n - 1) + 1)][Ca_{n,1} + (1 - C) a_{n,2}] b_n$$

$$= C \sum_{n=2}^{\infty} [(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n - 1) + 1)] a_{n,1} b_n$$

$$+ (1 - C) \sum_{n=2}^{\infty} [(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n - 1) + 1)] a_{n,2} b_n$$

$\le C(\alpha\beta - 2) + (1 - C)(\alpha\beta - 2)$ (since $f_1,\ f_2 \in AD(\lambda, \gamma, \alpha, \beta)$)

$\le C\alpha\beta - C2 + \alpha\beta - 2 - C\alpha\beta + C2$

$\le \alpha\beta - 2$

$\Rightarrow g \in AD(\lambda, \gamma, \alpha, \beta).$

4. Extreme Points

Theorem 4.1. *Let* $f_1(z) = z$ *and let*

$$f_n(z) = z + \frac{\alpha\beta - 2}{((n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n - 1) + 1)) b_n} z^n$$

$$n = 2, 3, \ldots.$$

Then $f \in AD(\lambda, \gamma, \alpha, \beta)$ *if and only if f can be expressed in the*

$$f(z) = \sum_{n=1}^{\infty} C_n f_n(z), \tag{4.1}$$

where $C_n \geq 0$ *and* $\sum_{n=1}^{\infty} C_n = 1.$

Proof. Suppose *f* can be expressed in the form (4.1), then

$$f(z) = \sum_{n=1}^{\infty} C_n f_n(z)$$

$$= \sum_{n=1}^{\infty} C_n \left[z + \frac{\alpha\beta - 2}{(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1)+1))b_n} z^n \right]$$

$$= z + \sum_{n=2}^{\infty} C_n \frac{\alpha\beta - 2}{(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1)+1)b_n} z^n.$$

Now

$$\sum_{n=2}^{\infty} [(n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1)+1)]b_n$$

$$C_n \frac{\alpha\beta - 2}{((n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1)+1))b_n}$$

$$= \sum_{n=2}^{\infty} C_n(\alpha\beta - 2)$$

$$= (\alpha\beta - 2) \sum_{n=2}^{\infty} C_n$$

$$= \alpha\beta - 2(1 - C_1)$$

$$\leq \alpha\beta - 2 \quad (\text{since } (1 - C_1) \leq 1)$$

which implies $f \in AD(\lambda, \gamma, \alpha, \beta)$.

Conversely: Suppose that $f \in AD(\lambda, \gamma, \alpha, \beta)$, then by Corollary 1.1

$$a_n \le \frac{\alpha\beta - 2}{((n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1) + 1))b_n} \quad n = 2, 3, \ldots.$$

Setting

$$C_n = \frac{\alpha\beta - 2}{((n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1) + 1))b_n} \quad n = 2, 3, \ldots.$$

and $C_1 = 1 - \sum_{n=2}^{\infty} C_n$.

We notice that $f(z) = \sum_{n=1}^{\infty} C_n f_n(z)$.

Hence the result.

5. Radius of Starlikeness and Convexity

In this section we derive the radii results for functions in the class $AD(\lambda, \gamma, \alpha, \beta)$ to be starlike or convex of order p.

Theorem 5.1. *If* $f \in AD(\lambda, \gamma, \alpha, \beta)$, *then f is univalent starlike function of order p,* $0 \le p \le 1$ *in the disk* $|z| < R$, *where*

$$R = \inf_n \left[\frac{(1-p)(((n^2 - n)(\lambda\gamma(2n - \alpha\beta) + 2(\lambda - \gamma)) + 2n - \alpha\beta((\lambda - \gamma)(n-1) + 1))b_n)}{(n-p)(\alpha\beta - 2)} \right]^{\frac{1}{n-1}}, \quad n = 2, 3, \ldots.$$

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1 - p.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{zf'(z)-f(z)}{f(z)}\right|$$

$$= \left|\frac{z+\sum_{n=2}^{\infty} na_n z^n - z - \sum_{n=2}^{\infty} a_n z^n}{z+\sum_{n=2}^{\infty} a_n z^n}\right|$$

$$= \left|\frac{\sum_{n=2}^{\infty} na_n z^{n-1} - \sum_{n=2}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} na_n |z|^{n-1} - \sum_{n=2}^{\infty} a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

The last expression must bounded by $1-p$ if

$$\frac{\sum_{n=2}^{\infty} na_n |z|^{n-1} - \sum_{n=2}^{\infty} a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n |z|^{n-1}} \leq 1-p$$

$$\frac{\sum_{n=2}^{\infty} (n-p)a_n |z|^{n-1}}{1-p} \leq 1.$$

Hence by Corollary 1.1, then the last inequality will be true if

$$\frac{(n-p)}{1-p}|z|^{n-1} \le \frac{((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma)) + 2n - \alpha\beta((\lambda-\gamma)(n-1)+1))b_n}{\alpha\beta-2}$$

$$|z| \le \left[\frac{(1-p)(((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma)) + 2n - \alpha\beta((\lambda-\gamma)(n-1)+1))b_n)}{(n-p)(\alpha\beta-2)}\right]^{\frac{1}{n-1}}.$$

Let $|z| = R$. That is, the radius of starlikeness of order p for functions in the class $AD(\lambda, \gamma, \alpha, \beta)$.

Theorem 5.2. *If* $f \in AD(\lambda, \gamma, \alpha, \beta)$, *then f is univalent convex function of order p*, $0 \le p \le 1$ *in the disk* $|z| < R$, *where*

$$R = \inf_n \left[\frac{(1-p)(((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma)) + 2n - \alpha\beta((\lambda-\gamma)(n-1)+1))b_n)}{n(n-p)(\alpha\beta-2)}\right]^{\frac{1}{n-1}}, \quad n = 2, 3, \ldots.$$

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-p.$$

thus

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}}.$$

The last expression must bounded by $1-p$

$$\frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}} \le 1-p,$$

$$\frac{\sum_{n=2}^{\infty} n(n-p)a_n |z|^{n-1}}{1-p} \le 1.$$

Hence by Corollary 1.1, then the last inequality will be true if

$$\frac{n(n-p)}{1-p}|z|^{n-1} \le \frac{\begin{array}{r}((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma)) \\ +2n-\alpha\beta((\lambda-\gamma)(n-1)+1))b_n\end{array}}{\alpha\beta-2},$$

$$|z| \le \left[\frac{\begin{array}{r}(1-p)(((n^2-n)(\lambda\gamma(2n-\alpha\beta)+2(\lambda-\gamma)) \\ +2n-\alpha\beta((\lambda-\gamma)(n-1)+1))b_n)\end{array}}{n(n-p)(\alpha\beta-2)}\right]^{\frac{1}{n-1}}.$$

Let $|z| = R$. That is, the radius of convexity of order p for functions in the class $AD(\lambda, \gamma, \alpha, \beta)$.

6. Growth and Distortion Theorem

Theorem 6.1. *If the function* $f \in AD(\lambda, \gamma, \alpha, \beta)$, *then*

$$|z| - \frac{(\alpha\beta-2)}{[2(\lambda\gamma(4-\alpha\beta)+2(\lambda-\gamma))+4-\alpha\beta((\lambda-\gamma)+1)]b_2}|z|^2$$

$$\le |f(z)|$$

$$\le |z| + \frac{(\alpha\beta-2)}{[2(\lambda\gamma(4-\alpha\beta)+2(\lambda-\gamma))+4-\alpha\beta((\lambda-\gamma)+1)]b_2}|z|^2.$$

The result is sharp for

$$f(z) = z + \frac{\alpha\beta - 2}{[2(\lambda\gamma(4-\alpha\beta) + 2(\lambda-\gamma)) + 4 - \alpha\beta((\lambda-\gamma)+1)]b_2} z^2.$$

Proof. We have

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then

$$|f(z)| \le |z| + \sum_{n=2}^{\infty} a_n |z|^n$$

$$\le |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\le |z| + |z|^2 \frac{\alpha\beta - 2}{[2(\lambda\gamma(4-\alpha\beta) + 2(\lambda-\gamma)) + 4 - \alpha\beta((\lambda-\gamma)+1)]b_2}. \quad (6.1)$$

Similarly

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^n$$

$$\ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\ge |z| - |z|^2 \frac{\alpha\beta - 2}{[2(\lambda\gamma(4-\alpha\beta) + 2(\lambda-\gamma)) + 4 - \alpha\beta((\lambda-\gamma)+1)]b_2}. \quad (6.2)$$

Combining (6.1) and (6.2) we get

$$|z| - \frac{(\alpha\beta - 2)}{[2(\lambda\gamma(4-\alpha\beta) + 2(\lambda-\gamma)) + 4 - \alpha\beta((\lambda-\gamma)+1)]b_2} |z|^2$$

$$\le |f(z)|$$

$$\le |z| + \frac{(\alpha\beta - 2)}{[2(\lambda\gamma(4-\alpha\beta) + 2(\lambda-\gamma)) + 4 - \alpha\beta((\lambda-\gamma)+1)]b_n} |z|^2.$$

Theorem 6.2. *If the function* $f \in AD(\lambda, \gamma, \alpha, \beta)$, *then*

$$1 - \frac{2(\alpha\beta - 2)}{[2(\lambda\gamma(4 - \alpha\beta) + 2(\lambda - \gamma)) + 4 - \alpha\beta((\lambda - \gamma) + 1)]b_2} |z|$$

$$\leq |f'(z)|$$

$$\leq 1 + \frac{2(\alpha\beta - 2)}{[2(\lambda\gamma(4 - \alpha\beta) + 2(\lambda - \gamma)) + 4 - \alpha\beta((\lambda - \gamma) + 1)]b_2} |z|.$$

The result is sharp for

$$f(z) = z + \frac{\alpha\beta - 2}{[2(\lambda\gamma(4 - \alpha\beta) + 2(\lambda - \gamma)) + 4 - \alpha\beta((\lambda - \gamma) + 1)]b_2} z^2.$$

References

[1] M. Darus, Some subclasses of analytic functions, *J. Inst. Math. Comput. Sci. Math Ser.* 16(3) (2003), 121-126.

[2] A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.* 155 (1991), 364-370. https://doi.org/10.1016/0022-247X(91)90006-L

[3] T. Rosy, Studies on subclasses of starlike and convex functions, Ph.D. Thesis, University of Madras, 2001.

[4] S. Sunil Varma and Thomas Rosy, Certain properties of a subclass of univalent functions with finitely many fixed coefficients, *Khayyam J. Math.* 3(1) (2017), 25-32.

[5] K. A. Jassim, Some geometric properties of analytic functions associated with hypergeometric functions, *Iraqi Journal of Science* 57(1C) (2016), 705-712.