Some New Kulli-Basava Topological Indices

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Abstract

Recently, Kulli-Basava indices were introduced and studied their mathematical and chemical properties which have good response with mean isomer degeneracy. In this paper, we introduce the modified first and second Kulli-Basava indices, $F_1$-Kulli-Basava index, square Kulli-Basava index of a graph, and compute exact formulas for regular graphs, wheels, gear graphs and helm graphs.

1. Introduction

Throughout this paper $G$ is a finite, simple, connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(v)$ of a vertex $v$ is the number of vertices adjacent to $v$. Let $|V(G)| = n$ and $|E(G)| = m$. The degree of an edge $e = uv$ in $G$ is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. Let $S_e(v)$ denote the sum of degrees of all edges incident to a vertex $v$. We refer to [1] for undefined term and notation.

Recently, the first and second Kulli-Basava indices were introduced in [2], defined as

$$KB_1(G) = \sum_{uv \in E(G)} [S_e(u) + S_e(v)], \quad KB_2(G) = \sum_{uv \in E(G)} S_e(u)S_e(v).$$
We introduce the modified first and second Kulli-Basava indices, defined as

\[ mKB_1(G) = \sum_{u \in V(G)} \frac{1}{S_e(u) + S_e(v)}, \quad mKB_2(G) = \sum_{u \in V(G)} \frac{1}{S_e(u)S_e(v)}. \]

In [3], Furtula and Gutman studied the F-index, defined as

\[ F(G) = \sum_{u \in E(G)} [S_e(u)^2 + S_e(v)^2]. \]

Recently, the square ve-degree index was introduced by Kulli [4], defined as

\[ Q_{ve}(G) = \sum_{u \in E(G)} [d_{ve}(u) - d_{ve}(v)]^2. \]

We now propose the \( F_1 \)-Kulli-Basava and square Kulli-Basava indices, defined as

\[ F_1KB(G) = \sum_{u \in E(G)} [S_e(u)^2 + S_e(v)^2], \]
\[ QKB(G) = \sum_{u \in E(G)} [S_e(u) - S_e(v)]^2. \]

Recently, some \( F \)-indices were studied, for example, in [5, 6, 7, 8, 9, 10, 11] and also some square indices were studied, for example, in [12, 13, 14, 15, 16].

We introduce the \( F_1 \)-Kulli-Basava polynomial and square Kulli-Basava polynomial of a graph, defined as

\[ F_1KB(G, x) = \sum_{u \in E(G)} x^{S_e(u)^2 + S_e(v)^2}, \]
\[ QKB(G, x) = \sum_{u \in E(G)} x^{[S_e(u) - S_e(v)]^2}. \]

In this paper, we establish explicit formulas for the modified first and second Kulli-Basava indices, \( F_1 \)-Kulli-Basava and square Kulli-Basava indices of some graphs. Also the \( F_1 \)-Kulli-Basava and square Kulli-Basava polynomials of some graphs are obtained.
2. Regular Graphs

**Theorem 1.** Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then

\[(i) \ mKB_1(G) = \frac{m}{4r(r-1)}.\]  
\[(ii) \ mKB_2(G) = \frac{m}{4r^2(r-1)^2}.\]  
\[(iii) \ F_1KB(G) = 8mr^2(r-1)^2.\]  
\[(iv) \ QKB(G) = 0.\]

**Proof.** Let $G$ be an $r$-regular graph with $n$ vertices. Then $S_e(u) = 2r(u-1)$ for any vertex $u$ in $G$.

Thus

\[(i) \ mKB_1(G) = \sum_{uv \in E(G)} \frac{1}{S_e(u) + S_e(v)} = \frac{m}{2r(r-1) + 2r(r-1)} = \frac{m}{4r(r-1)}.\]

\[(ii) \ mKB_2(G) = \sum_{uv \in E(G)} \frac{1}{S_e(u)S_e(v)} = \frac{m}{2r(r-1)2r(r-1)} = \frac{m}{4r^2(r-1)^2}.\]

\[(iii) \ F_1KB(G) = \sum_{uv \in E(G)} [S_e(u)^2 + S_e(v)^2] = m[(2r(r-1))^2 + (2r(r-1))^2]\]

\[= 8mr^2(r-1)^2.\]

\[(iv) \ QKB(G) = \sum_{uv \in E(G)} (S_e(u) - S_e(v))^2 = 0.\]

**Corollary 1.1.** If $C_n$ is a cycle with $n$ vertices, then

\[(i) \ mKB_1(C_n) = \frac{n}{8}.\]  
\[(ii) \ mKB_2(C_n) = \frac{n}{16}.\]  
\[(iii) \ F_1KB(C_n) = 32n.\]  
\[(iv) \ QKB(C_n) = 0.\]

**Corollary 1.2.** If $K_n$ is a complete graph with $n$ vertices, then

\[(i) \ mKB_1(K_n) = \frac{n}{8(n-2)}.\]  
\[(ii) \ mKB_2(K_n) = \frac{n}{8(n-1)(n-2)^2}.\]  
\[(iii) \ F_1KB(K_n) = 4n(n-1)^3(n-2)^2.\]  
\[(iv) \ QKB(K_n) = 0.\]
Theorem 2. If \( G \) is an \( r \)-regular graph with \( n \) vertices and \( m \) edges, then

(i) \( F_1KB(G, x) = mx \, S_r^2(r-1)^2 \).  
(ii) \( QKB(G, x) = mx^0 \).

Proof. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then \( S_e(u) = 2r(r-1) \) for \( u \in V(G) \). Thus

(i) \( F_1KB(G, x) = \sum_{uv \in E(G)} x^{S_e(u)^2 + S_e(v)^2} = mx \, (2r(r-1))^2 = mx \, S_r^2(r-1)^2 \).

(ii) \( QKB(G, x) = \sum_{uv \in E(G)} x \, |S_e(u) - S_e(v)|^2 = mx^0 \).

Corollary 2.1. If \( C_n \) is a cycle with \( n \) vertices, then

(i) \( F_1KB(C_n, x) = nx^{32} \).  
(ii) \( QKB(C_n, x) = nx^0 \).

Corollary 2.2. If \( K_n \) is a complete graph with \( n \) vertices, then

(i) \( F_1KB(K_n, x) = \frac{n(n-1)}{2} \, x^{8(n-1)^2(n-2)^2} \).  
(ii) \( QKB(K_n, x) = \frac{n(n-1)}{2} \, x^0 \).

3. Wheel Graphs

A wheel \( W_n \) is the join of \( K_1 \) and \( C_n \). Clearly \( W_n \) has \( n + 1 \) vertices and \( 2n \) edges. A wheel \( W_n \) is presented in Figure 1. The vertices of \( C_n \) are called rim vertices and the vertex of \( K_1 \) is called apex.

![Figure 1. Wheel \( W_n \).](http://www.earthlinepublishers.com)
Lemma 1. Let $W_n$ be a wheel with $n + 1$ vertices and $2n$ edges, $n \geq 3$. Then

$$E_1 = \{uv \in E(W_n) | S_e(u) = n + 9, (S_e(v) = n(n + 1))\}, \quad |E_1| = n$$

$$E_2 = \{uv \in E(W_n) | S_e(u) = n + 9, (S_e(v) = n + 9)\}, \quad |E_2| = n.$$ 

Theorem 3. Let $W_n$ be a wheel with $n + 1$ vertices and $2n$ edges, $n \geq 3$. Then

(i) $m_{KB_1}(W_n) = \frac{n}{n^2 + 2n + 9} + \frac{n}{2n + 18}.$

(ii) $m_{KB_2}(W_n) = \frac{1}{(n + 9)(n + 1)} + \frac{n}{(n + 9)^2}.$

(iii) $F_1KB(W_n) = (n^3 + 5n^2 + 55n + 243).$

(iv) $QKB(W_n) = n(n^2 - 9)^2.$

Proof. By using definitions and Lemma 1, we derive

(i) $m_{KB_1}(W_n) = \sum_{uv \in E(W_n)} \frac{1}{S_e(u) + S_e(v)}$

$$= |E_1| \left(\frac{1}{n + 9 + n(n + 1)}\right) + |E_2| \left(\frac{1}{n + 9 + n + 9}\right)$$

$$= \frac{n}{n^2 + 2n + 9} + \frac{n}{2n + 18}.$$

(ii) $m_{KB_2}(W_n) = \sum_{uv \in E(W_n)} \frac{1}{S_e(u)S_e(v)}$

$$= |E_1| \left(\frac{1}{(n + 9) \times n(n + 1)}\right) + |E_2| \left(\frac{1}{(n + 9)(n + 9)}\right)$$

$$= \frac{1}{(n + 9)(n + 1)} + \frac{n}{(n + 9)^2}.$$

(iii) $F_1KB(W_n) = \sum_{uv \in E(W_n)} [S_e(u)^2 + S_e(v)^2]$
Theorem 4. Let $W_n$ be a wheel with $n + 1$ vertices and $2n$ edges, $n \geq 3$. Then

(i) $F_1KB(W_n, x) = nx^{n^3 + 3n^2 + 19n + 81} + nx^{2(n+9)^2}$.

(ii) $QKB(W_n, x) = nx^{(n^2-9)^2} + nx^0$.

Proof. By using definitions and Lemma 1, we deduce

(i) $F_1KB(W_n, x) = \sum_{uv \in E(W_n)} x^{[S_e(u) + S_e(v)]^2}$

$$= |E_1| x^{(n+9)^2 + n^2(n+1)^2} + |E_2| x^{(n+9)^2 + (n+9)^2}$$

$$= nx^{n^3 + 3n^2 + 19n + 81} + nx^{2(n+9)^2}.$$ 

(ii) $QKB(W_n, x) = \sum_{uv \in E(W_n)} x^{[S_e(u) - S_e(v)]^2}$

$$= |E_1| x^{(n+9)(n+9(n+1))} + |E_2| x^{(n+9)(n+9(n+1))}$$

$$= nx^{(n^2-9)^2} + nx^0.$$ 

4. Gear Graphs

A graph is a gear graph obtained from $W_n$ by adding a vertex between each pair of adjacent rim vertices and it is denoted by $G_n$. Clearly $G_n$ has $2n + 1$ vertices and $3n$ edges. A graph $G_n$ is depicted in Figure 2.
Lemma 2. Let $G_n$ be a gear graph with $3n$ edges. Then $G_n$ has two types of edges as follows:

$$E_1 = \{uv \in E(G_n) | S_e(u) = n(n + 1), S_e(v) = n + 7\}, \quad |E_1| = n.$$  
$$E_2 = \{uv \in E(G_n) | S_e(u) = n + 7, S_e(v) = 6\}, \quad |E_2| = 2n.$$ 

Theorem 5. If $G_n$ is a gear graph with $2n + 1$ vertices and $3n$ edges, then

(i) $^m KB_1(G_n) = \frac{n}{n^2 + 2n + 7} + \frac{2n}{n + 13}.$

(ii) $^m KB_2(G_n) = \frac{1}{(n + 1)(n + 7)} + \frac{n}{3(n + 7)}.$

(iii) $F_1KB(G_n) = n(n^4 + 2n^3 + 4n^2 + 42n + 219).$

(iv) $QKB(G_n) = n(n^4 - 12n^2 + 4n + 51).$

Proof. By using definitions and Lemma 2, we deduce

(i) $^m KB_1(G_n) = \sum_{uv \in E(G_n)} \frac{1}{S_e(u) + S_e(v)}$

$$= |E_1| \left( \frac{1}{n(n + 1) + (n + 7)} \right) + |E_2| \left( \frac{1}{n + 7 + 6} \right)$$

$$= \frac{n}{n^2 + 2n + 7} + \frac{2n}{n + 13}.$$
Theorem 6. Let $G_n$ be a gear graph with $2n + 1$ vertices and $3n$ edges, $n \geq 3$. Then

(i) $F_1 KB(G_n, x) = nx^{n^2 + 3n^2 + 15n + 49} + 2nx^{n^2 + 14n + 85}$.

(ii) $QKB(G_n, x) = nx^{(n^2 - 7)^2} + 2nx^{(n+1)^2}$.

Proof. By using definitions and Lemma 2, we obtain

(i) $F_1 KB(G_n, x) = \sum_{uv \in E(G_n)} x[S_e(u) + S_e(v)]$

$$= |E_1| \left(\frac{1}{n(n+1)(n+7)}\right) + |E_2| \left(\frac{1}{(n+7)(6)}\right)$$

$$= \frac{1}{n(n+1)(n+7) + \frac{n}{3(n+7)}}.$$

(ii) $QKB(G_n, x) = \sum_{uv \in E(G_n)} [S_e(u) + S_e(v)]$

$$= |E_1| \left(\frac{(n^2 + n)^2 + (n+7)^2}{2} + |E_2| \left(\frac{(n+7)^2 + 6^2}{2}$$

$$= n(n^4 + 2n^3 + 4n^2 + 42n + 219).$$

$$= n(n^4 - 12n^2 + 4n + 51).$$
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(ii) $QKB(G_n, x) = \sum_{uv \in E(G_n)} x^{|S_e(u) - S_e(v)|^2}

= |E_1| x^{(n^2+n-n-7)^2} + |E_2| x^{(n+7-6)^2}

= nx^{(n^2-7)^2} + 2nx^{(n+1)^2}.

5. Helm Graphs

A helm graph $H_n$ is a graph obtained from $W_n$ by attaching an end edge to each rim vertex. Clearly $H_n$ has $2n + 1$ vertices and $3n$ edges. A graph $H_n$ is shown in Figure 3.

![Figure 3. Helm graph $H_n$.](image)

Lemma 3. Let $H_n$ be a helm graph with $3n$ edges. Then $H_n$ has three types of edges as given below:

$E_1 = \{uv \in E(H_n) | S_e(u) = n(n + 2), S_e(v) = n + 17\}, \quad |E_1| = n.$

$E_2 = \{uv \in E(H_n) | S_e(u) = S_e(v) = n + 17\}, \quad |E_2| = n.$

$E_3 = \{uv \in E(H_n) | S_e(u) = n + 17, S_e(v) = 3\}, \quad |E_3| = n.$

Theorem 7. Let $H_n$ be a helm graph with $2n + 1$ vertices and $3n$ edges. Then

(i) $m KB_1(H_n) = \frac{n}{n^2 + 3n + 17} + \frac{n}{2n + 34} + \frac{n}{n + 20}.$
\( (ii) \quad m \text{KB}_2(H_n) = \frac{1}{(n + 2)(n + 17)} + \frac{n}{(n + 17)^2} + \frac{n}{3(n + 17)}. \)

\( (iii) \quad FKB(H_n) = n[n^2(n + 2)^2 + (n + 17)^2] + 2n(n + 17)^2 + n[(n + 17)^2 + 9]. \)

\( (iv) \quad QKB(H_n) = n(n^2 + n - 17)^2 + n(n + 14)^2. \)

**Proof.** By using definitions and Lemma 3, we deduce

\( (i) \quad m \text{KB}_1(H_n) = \sum_{uv \in E(H_n)} \frac{1}{S_e(u) + S_e(v)} \)

\[ = |E_1| \left( \frac{1}{n(n + 2) + n + 17} \right) + |E_2| \left( \frac{1}{n + 17 + n + 17} \right) \]

\[ + |E_3| \left( \frac{1}{n + 17 + 3} \right) \]

\[ = \frac{n}{n^2 + 3n + 17} + \frac{n}{2n + 34} + \frac{n}{n + 20}. \]

\( (ii) \quad m \text{KB}_2(H_n) = \sum_{uv \in E(H_n)} \frac{1}{S_e(u)S_e(v)} \)

\[ = |E_1| \left( \frac{1}{n(n + 2)(n + 17)} \right) + |E_2| \left( \frac{1}{(n + 17)(n + 17)} \right) \]

\[ + |E_3| \left( \frac{1}{(n + 17)3} \right) \]

\[ = \frac{1}{(n + 2)(n + 17)} + \frac{n}{(n + 17)^2} + \frac{n}{3(n + 17)}. \]

\( (iii) \quad F_1KB(H_n) = \sum_{uv \in E(H_n)} [S_e(u)^2 + S_e(v)^2] \)

\[ = |E_1| [n^2(n + 2)^2 + (n + 17)^2] + |E_2| [(n + 17)^2 + (n + 17)^2] \]

\[ + |E_3| [(n + 17)^2 + 3^2] \]

\[ = n[n^2(n + 2)^2 + (n + 17)^2] + 2n(n + 17)^2 + n[(n + 17)^2 + 9] \]
(iv) \( QKB(H_n) = \sum_{uv \in E(H_n)} [S_e(u) - S_e(v)]^2 \)

\[ = |E_1| \left( n^2 + 2n - n - 17 \right)^2 + |E_2| \left( n + 17 - n - 17 \right)^2 \]

\[ + |E_3| \left( n + 17 - 3 \right)^2 \]

\[ = n(n^2 + n - 17) + n(n + 14)^2. \]

**Theorem 8.** Let \( H_n \) be a helm graph with \( 2n + 1 \) vertices and \( 3n \) edges. Then

(i) \( F_1 KB(H_n, x) = nx^{n^2(n+2)^2+(n+17)^2} + nx^{2(n+17)^2} + nx^{(n+17)^2+9}. \)

(ii) \( QKB(H_n, x) = nx^{(n^2+n-17)^2} + nx^0 + nx^{(n+14)^2}. \)

**Proof.** By using definitions and Lemma 3, we obtain

(i) \( F_1 KB(H_n, x) = \sum_{uv \in E(H_n)} x\left[ S_e(u)^2 + S_e(v)^2 \right] \)

\[ = |E_1| x^{n^2(n+2)^2+(n+17)^2} + |E_2| x^{(n+17)^2+(n+17)^2} \]

\[ + |E_3| x^{(n+17)^2+3^2} \]

\[ = nx^{n^2(n+2)^2+(n+17)^2} + nx^{2(n+17)^2} + nx^{(n+17)^2+9}. \]

(ii) \( QKB(H_n, x) = \sum_{uv \in E(H_n)} x\left[ S_e(u)^2 - S_e(v)^2 \right] \)

\[ = |E_1| x^{(n^2+2n-n-17)^2} + |E_2| x^{(n+17-n-17)^2} + |E_3| x^{(n+17-3)^2} \]

\[ = nx^{(n^2+n-17)^2} + nx^0 + nx^{(n+14)^2}. \]

**References**


