Strong Insertion of a Contra-$\alpha$-continuous Function between Two Comparable Real-valued Functions

Majid Mirmiran$^1$ and Binesh Naderi$^2$

$^1$Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
  e-mail: mirmir@sci.ui.ac.ir

$^2$School of Management and Medical Information, Medical University of Isfahan, Iran
  e-mail: naderi@mng.mui.ac.ir

Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a contra-$\alpha$-continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour et al. [21], while the concept of a, locally dense, set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [18]. A subset $A$ of a topological space $(X, \tau)$ is called semi-open [10] if...
$A \subseteq \text{Cl(Int}(A))$. A set $A$ is called semi-closed if its complement is semi-open or equivalently if $\text{Int(Cl}(A)) \subseteq A$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open if $A$ is the difference of an open and a nowhere dense subset of $X$. A set $A$ is called $\alpha$-closed if its complement is $\alpha$-open or equivalently if $A$ is union of a closed and a nowhere dense set.

We have a set is $\alpha$-open if and only if it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [20]. He investigated the sets that can be represented as union of closed sets and called them $V$-sets. Complements of $V$-sets, i.e., sets that are intersection of open sets are called $\Lambda$-sets [20].

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$-continuous [25] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. Jafari and Noiri in [12, 13] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha$-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 23].

Hence, a real-valued function $f$ defined on a topological space $X$ is called contra-$\alpha$-continuous (resp. contra-semi-continuous, contra-precontinuous) if the preimage of every open subset of $\mathbb{R}$ is $\alpha$-closed (resp. semi-closed, preclosed) in $X$ [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-$\alpha$-continuous function between two comparable real-valued functions.

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all $x$ in $X$.

The following definitions are modifications of conditions considered in [16].
A property $P$ defined relative to a real-valued function on a topological space is a $c\alpha$-property provided that any constant function has property $P$ and provided that the sum of a function with property $P$ and any contra-$\alpha$-continuous function also has property $P$. If $P_1$ and $P_2$ are $c\alpha$-property, the following terminology is used: (i) A space $X$ has the weak $c\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-$\alpha$-continuous function $h$ such that $g \leq h \leq f$. (ii) A space $X$ has the strong $c\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-$\alpha$-continuous function $h$ such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose $\alpha$-kernel of sets are $\alpha$-open, is given a sufficient condition for the weak $c\alpha$-insertion property. Also for a space with the weak $c\alpha$-insertion property, we give necessary and sufficient conditions for the space to have the strong $c\alpha$-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-$\alpha$-continuous function, the necessary definitions and terminology are stated.

The abbreviations $c\alpha c$, $cpc$ and $csc$ are used for contra-$\alpha$-continuous, contra-precontinuous and contra-semi-continuous, respectively.

Let $(X, \tau)$ be a topological space. Then the family of all $\alpha$-open, $\alpha$-closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

**Definition 2.1.** Let $A$ be a subset of a topological space $(X, \tau)$. We define the subsets $A^\Lambda$ and $A^V$ as follows:

$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.

In [7, 19, 22], $A^\Lambda$ is called the kernel of $A$. 

We define the subsets $\alpha(A^\Lambda)$, $\alpha(A^V)$, $p(A^\Lambda)$, $p(A^V)$, $s(A^\Lambda)$ and $s(A^V)$ as follows:

\[
\alpha(A^\Lambda) = \bigcap\{O : O \supseteq A, O \in \alpha O(X, \tau)\},
\]

\[
\alpha(A^V) = \bigcup\{F : F \subseteq A, F \in \alpha C(X, \tau)\},
\]

\[
p(A^\Lambda) = \bigcap\{O : O \supseteq A, O \in p O(X, \tau)\},
\]

\[
p(A^V) = \bigcup\{F : F \subseteq A, F \in p C(X, \tau)\},
\]

\[
s(A^\Lambda) = \bigcap\{O : O \supseteq A, O \in s O(X, \tau)\},
\]

\[
s(A^V) = \bigcup\{F : F \subseteq A, F \in s C(X, \tau)\}.
\]

$\alpha(A^\Lambda)$ (resp. $p(A^\Lambda)$, $s(A^\Lambda)$) is called the $\alpha$-kernel (resp. prekernel, semi-kernel) of $A$.

The following first two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If $\rho$ is a binary relation in a set $S$, then $\bar{\rho}$ is defined as follows:

$x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any $u$ and $v$ in $S$.

**Definition 2.3.** A binary relation $\rho$ in the power set $P(X)$ of a topological space $X$ is called a strong binary relation in $P(X)$ in case $\rho$ satisfies each of the following conditions:

1. If $A_i \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set $C$ in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2. If $A \subseteq B$, then $A \bar{\rho} B$.

3. If $A \rho B$, then $\alpha(A^\Lambda) \subseteq B$ and $A \subseteq \alpha(B^V)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is called a lower indefinite cut set in the domain of $f$ at the level $\ell$. 

http://www.earthlinepublishers.com
We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), in which \( \alpha \)-kernel sets are \( \alpha \)-open, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \), then \( A(f, t_1) \rho A(g, t_2) \), then there exists a contra-\( \alpha \)-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on the \( X \) such that \( g \leq f \). By hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \), then \( A(f, t_1) \rho A(g, t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( \mathbb{Q} \) into the power set of \( X \) by \( F(t) = A(f, t) \) and \( G(t) = A(g, t) \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then \( F(t_1) \rho F(t_2), G(t_1) \rho G(t_2) \), and \( F(t_1) \rho G(t_2) \). By Lemmas 1 and 2 of [15] it follows that there exists a function \( H \) mapping \( \mathbb{Q} \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \rho H(t_1), H(t_1) \rho H(t_2) \) and \( H(t_1) \rho G(t_2) \).

For any \( x \) in \( X \), let \( h(x) = \inf \{ t \in \mathbb{Q} : x \in H(t) \} \).

We first verify that \( g \leq h \leq f \): If \( x \) is in \( H(t) \), then \( x \) is in \( G(t') \) for any \( t' > t \); since \( x \) is in \( G(t') = A(g, t') \) implies that \( g(x) \leq t' \), it follows that \( g(x) \leq t \). Hence \( g \leq h \). If \( x \) is not in \( H(t) \), then \( x \) is not in \( F(t') \) for any \( t' < t \); since \( x \) is not in \( F(t') = A(f, t') \) implies that \( f(x) > t' \), it follows that \( f(x) \geq t \). Hence \( h \leq f \).

Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1, t_2) = \alpha(H(t_2)^\mathbb{V} \setminus \alpha(H(t_1)) \setminus) \). Hence \( h^{-1}(t_1, t_2) \) is \( \alpha \)-closed in \( X \), i.e., \( h \) is a contra-\( \alpha \)-continuous function on \( X \).

The above proof used the technique of Theorem 1 in [14].

If a space has the strong \( c\alpha \)-insertion property for \( (P_1, P_2) \), then it has the weak
$c\alpha$-insertion property for $(P_1, P_2)$. The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $c\alpha$-insertion property to satisfy the strong $c\alpha$-insertion property.

**Theorem 2.2.** Let $P_1$ and $P_2$ be $c\alpha$-property and $X$ be a space that satisfies the weak $c\alpha$-insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the strong $c\alpha$-insertion property for $(P_1, P_2)$ if and only if there exist lower cut sets $A(f - g, 2^{-n})$ and there exists a sequence \( \{F_n\} \) of subsets of $X$ such that (i) for each $n$, $F_n$ and $A(f - g, 2^{-n})$ are completely separated by contra-$\alpha$-continuous functions, and (ii) \( \{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n \).

**Proof.** Suppose that there is a sequence \( (A(f - g, 2^{-n})) \) of lower cut sets for $f - g$ and suppose that there is a sequence \( (F_n) \) of subsets of $X$ such that

\[
\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n
\]

and such that for each $n$, there exists a contra-$\alpha$-continuous function $k_n$ on $X$ into \([0, 2^{-n}]\) with $k_n = 2^{-n}$ on $F_n$ and $k_n = 0$ on $A(f - g, 2^{-n})$. The function $k$ from $X$ into \([0, 1/4]\) which is defined by

\[
k(x) = \frac{1}{4} \sum_{n=1}^{\infty} k_n(x)
\]

is a contra-$\alpha$-continuous function by the Cauchy condition and the properties of contra-$\alpha$-continuous functions, (1) $k^{-1}(0) = \{x \in X : (f - g)(x) = 0\}$ and (2) if $(f - g)(x) > 0$, then $k(x) < (f - g)(x)$: In order to verify (1), observe that if $(f - g)(x) = 0$, then $x \in A(f - g, 2^{-n})$ for each $n$ and hence $k_n(x) = 0$ for each $n$. Thus $k(x) = 0$. Conversely, if $(f - g)(x) > 0$, then there exists an $n$ such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that
Strong Insertion of a Contra-\(\alpha\)-continuous Function …

\[
\{x \in X : (f - g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f - g, 2^{-n})
\]

and that \((A(f - g, 2^{-n}))\) is a decreasing sequence. Thus if \((f - g)(x) > 0\), then either \(x \notin A(f - g, 1/2)\) or there exists a smallest \(n\) such that \(x \notin A(f - g, 2^{-n})\) and \(x \in A(f - g, 2^{-j})\) for \(j = 1, \ldots, n - 1\).

In the former case,

\[
k(x) = \frac{1}{4} \sum_{n=1}^{\infty} k_n(x) \leq \frac{1}{4} \sum_{n=1}^{\infty} 2^{-n} < \frac{1}{2} \leq (f - g)(x),
\]

and in the latter,

\[
k(x) = \frac{1}{4} \sum_{j=n}^{\infty} k_j(x) \leq \frac{1}{4} \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f - g)(x).
\]

Thus \(0 \leq k \leq f - g\) and if \((f - g)(x) > 0\), then \((f - g)(x) > k(x) > 0\). Let \(g_1 = g + (1/4)k\) and \(f_1 = f - (1/4)k\). Then \(g \leq g_1 \leq f_1 \leq f\) and if \(g(x) < f(x)\), then

\[
g(x) < g_1(x) < f_1(x) < f(x).
\]

Since \(P_1\) and \(P_2\) are \(c\alpha\)-properties, \(g_1\) has property \(P_1\) and \(f_1\) has property \(P_2\). Since by hypothesis \(X\) has the weak \(c\alpha\)-insertion property for \((P_1, P_2)\), there exists a contra-\(\alpha\)-continuous function \(h\) such that \(g_1 \leq h \leq f_1\). Thus \(g \leq h \leq f\) and if \(g(x) < f(x)\), then \(g(x) < h(x) < f(x)\). Therefore \(X\) has the strong \(c\alpha\)-insertion property for \((P_1, P_2)\). (The technique of this proof is by Lane [16].)

Conversely, assume that \(X\) satisfies the strong \(c\alpha\)-insertion for \((P_1, P_2)\). Let \(g\) and \(f\) be functions on \(X\) satisfying \(P_1\) and \(P_2\), respectively such that \(g \leq f\). Thus, there exists a contra-\(\alpha\)-continuous function \(h\) such that \(g \leq h \leq f\) and such that if \(g(x) < f(x)\) for any \(x\) in \(X\), then \(g(x) < h(x) < f(x)\). We follow an idea contained in Powderly [24]. Now consider the functions \(0\) and \(f - h\). \(0\) satisfies property \(P_1\) and \(f - h\) satisfies

---

Thus, there exists a contra-\(\alpha\)-continuous function \(h_1\) such that \(0 \leq h_1 \leq f - h\) and if \(0 < (f - h)(x)\) for any \(x\) in \(X\), then \(0 < h_1(x) < (f - h)(x)\). We next show that

\[
\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.
\]

If \(x\) is such that \((f - g)(x) > 0\), then \(g(x) < f(x)\). Therefore, \(g(x) < h(x) < f(x)\). Thus, \(f(x) - h(x) > 0\) or \((f - h)(x) > 0\). Hence, \(h_1(x) > 0\). On the other hand, if \(h_1(x) > 0\), then since \((f - h) \geq h_1\) and \(f - g \geq f - h\), therefore \((f - g)(x) > 0\). For each \(n\), let \(A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}\), and

\[
F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}
\]

and

\[
k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.
\]

Since \(\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}\), it follows that

\[
\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.
\]

We next show that \(k_n\) is a contra-\(\alpha\)-continuous function which completely separates \(F_n\) and \(A(f - g, 2^{-n})\). From its definition and by the properties of contra-\(\alpha\)-continuous functions, it is clear that \(k_n\) is a contra-\(\alpha\)-continuous function. Let \(x \in F_n\). Then, from the definition of \(k_n\), \(k_n(x) = 2^{-n}\). If \(x \in A(f - g, 2^{-n})\), then since \(h_1 \leq f - h \leq f - g\), \(h_1(x) \leq 2^{-n}\). Thus, \(k_n(x) = 0\), according to the definition of \(k_n\). Hence \(k_n\) completely separates \(F_n\) and \(A(f - g, 2^{-n})\).

**Theorem 2.3.** Let \(P_1\) and \(P_2\) be \(c\alpha\)-properties and assume that the space \(X\) satisfied the weak \(c\alpha\)-insertion property for \((P_1, P_2)\). The space \(X\) satisfies the strong \(c\alpha\)-insertion property for \((P_1, P_2)\) if and only if \(X\) satisfies the strong \(c\alpha\)-insertion property for \((P_1, c\alpha c)\) and for \((c\alpha c, P_2)\).

**Proof.** Assume that \(X\) satisfies the strong \(c\alpha\)-insertion property for \((P_1, c\alpha c)\) and for
(cαc, P₂). If g and f are functions on X such that , then since X satisfies the weak cα-insertion property for (P₁, P₂) there is a contra-α-continuous function k such that . Also, by hypothesis there exist contra-α-continuous functions h₁ and h₂ such that and if , then and such that and if , then . If a function h is defined by , then h is a contra-α-continuous function, and if , then . Hence X satisfies the strong cα-insertion property for (P₁, P₂).

The converse is obvious since any contra-α-continuous function must satisfy both properties P₁ and P₂. (The technique of this proof is by Lane [17].)

3. Applications

Before stating the consequences of Theorems 2.1, 2.2 and 2.3 we suppose that X is a topological space whose α-kernel sets are α-open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi-open) sets , of X, there exist α-closed sets F₁ and F₂ of X such that , and , then X has the weak cα-insertion property for (cpc, cpc) (resp. (csc, csc)).

**Proof.** Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and g ≤ f. If a binary relation ρ is defined by Ap B in case , then by hypothesis ρ is a strong binary relation in the power set of X. If and are any elements of Q with , then

since , is a preopen (resp. semi-open) set and since , is a preclosed (resp. semi-closed) set, it follows that , (resp. ). Hence t₁ < t₂ implies that \( A(f, t₁) \rho A(g, t₂) \). The proof follows from Theorem 2.1.

---

Corollary 3.2. If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \), there exist \( \alpha \)-closed sets \( F_1 \) and \( F_2 \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then every contra-precontinuous (resp. contra-semi-continuous) function is contra-\( \alpha \)-continuous.

Proof. Let \( f \) be a real-valued contra-precontinuous (resp. contra-semi-continuous) function defined on \( X \). Set \( g = f \), then by Corollary 3.1, there exists a contra-\( \alpha \)-continuous function \( h \) such that \( g = h = f \). 

Corollary 3.3. If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \) of \( X \), there exist \( \alpha \)-closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then \( X \) has the strong \( c\alpha \)-insertion property for \((cpc, cpc)\) (resp. \((csc, csc)\)).

Proof. Let \( g \) and \( f \) be real-valued functions defined on \( X \), such that \( f \) and \( g \) are \( cpc \) (resp. \( csc \)), and \( g \leq f \). Set \( h = (f + g)/2 \), thus \( g \leq h \leq f \) and if \( g(x) < f(x) \) for any \( x \) in \( X \), then \( g(x) < h(x) < f(x) \). Also, by Corollary 3.2, since \( g \) and \( f \) are contra-\( \alpha \)-continuous functions hence \( h \) is a contra-\( \alpha \)-continuous function.

Corollary 3.4. If for each pair of disjoint subsets \( G_1, G_2 \) of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi-open, there exist \( \alpha \)-closed subsets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then \( X \) have the weak \( c\alpha \)-insertion property for \((cpc, cpc)\) and \((csc, csc)\).

Proof. Let \( g \) and \( f \) be real-valued functions defined on \( X \), such that \( g \) is \( cpc \) (resp. \( csc \)) and \( f \) is \( csc \) (resp. \( cpc \)), with \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( s(A^\Lambda) \subseteq p(B^\Lambda) \) (resp. \( p(A^\Lambda) \subseteq s(B^\Lambda) \)), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);
\]

since \( \{x \in X : f(x) \leq t_1\} \) is a semi-open (resp. preopen) set and since \( \{x \in X : g(x) < t_2\} \) is a preclosed (resp. semi-closed) set, it follows that \( s(A(f, t_1)^\Lambda) \subseteq p(A(g, t_2)^\Lambda) \).
Strong Insertion of a Contra-α-continuous Function … 233

Hence, \( t_1 < t_2 \) implies that \( A(f, t_1) \supseteq A(g, t_2) \).

The proof follows from Theorem 2.1. \( \square \)

Before stating consequences of Theorems 2.2 and 2.3 we state and prove the necessary lemmas.

**Lemma 3.1.** The following conditions on the space \( X \) are equivalent:

(i) For each pair of disjoint subsets \( G_1, G_2 \) of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi-open, there exist \( \alpha \)-closed subsets \( F_1, F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \).

(ii) If \( G \) is a semi-open (resp. preopen) subset of \( X \) which is contained in a preclosed (resp. semi-closed) subset \( F \) of \( X \), then there exists an \( \alpha \)-closed subset \( H \) of \( X \) such that \( G \subseteq H \subseteq \alpha(H^\alpha) \subseteq F \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( G \subseteq F \), where \( G \) and \( F \) are semi-open (resp. preopen) and preclosed (resp. semi-closed) subsets of \( X \), respectively. Hence, \( F^c \) is a preopen (resp. semi-open) and \( G \cap F^c = \emptyset \).

By (i) there exists two disjoint \( \alpha \)-closed subsets \( F_1, F_2 \) such that \( G \subseteq F_1 \) and \( F^c \subseteq F_2 \). But

\[ F^c \subseteq F_2 \Rightarrow F_1^c \subseteq F, \]

and

\[ F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c \]

hence

\[ G \subseteq F_1 \subseteq F_2^c \subseteq F \]

and since \( F_2^c \) is an \( \alpha \)-open subset containing \( F_1 \), we conclude that \( \alpha(F_1^\alpha) \subseteq F_2^c \), i.e.,

\[ G \subseteq F_1 \subseteq \alpha(F_1^\alpha) \subseteq F. \]

By setting \( H = F_1 \), condition (ii) holds.

(ii) \( \Rightarrow \) (i) Suppose that \( G_1, G_2 \) are two disjoint subsets of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi-open.
This implies that \( G_2 \subseteq G_1^c \) and \( G_1^c \) is a preclosed subset of \( X \). Hence by (ii) there exists an \( \alpha \)-closed set \( H \) such that \( G_2 \subseteq H \subseteq \alpha(H^\Lambda) \subseteq G_1^c \).

But

\[
H \subseteq \alpha(H^\Lambda) \Rightarrow H \cap \alpha((H^\Lambda)^c) = \emptyset
\]

and

\[
\alpha(H^\Lambda) \subseteq G_1^c \Rightarrow G_1 \subseteq \alpha((H^\Lambda)^c).
\]

Furthermore, \( \alpha((H^\Lambda)^c) \) is an \( \alpha \)-closed subset of \( X \). Hence \( G_2 \subseteq H, G_1 \subseteq \alpha((H^\Lambda)^c) \) and \( H \cap \alpha((H^\Lambda)^c) = \emptyset \). This means that condition (i) holds.

**Lemma 3.2.** Suppose that \( X \) is a topological space. If each pair of disjoint subsets \( G_1, G_2 \) of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi-open, can be separated by \( \alpha \)-closed subsets of \( X \), then there exists a contra-\( \alpha \)-continuous function \( h : X \to [0, 1] \) such that \( h(G_2) = \{0\} \) and \( h(G_1) = \{1\} \).

**Proof.** Suppose \( G_1 \) and \( G_2 \) are two disjoint subsets of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi-open. Since \( G_1 \cap G_2 = \emptyset \), hence \( G_2 \subseteq G_1^c \). In particular, since \( G_1^c \) is a preclosed subset of \( X \) containing the semi-open subset \( G_2 \) of \( X \), by Lemma 3.1, there exists an \( \alpha \)-closed subset \( H_{1/2} \) such that

\[
G_2 \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^\Lambda) \subseteq G_1^c.
\]

Note that \( H_{1/2} \) is also a preclosed subset of \( X \) and contains \( G_2 \), and \( G_1^c \) is a preclosed subset of \( X \) and contains the semi-open subset \( \alpha(H_{1/2}^\Lambda) \) of \( X \). Hence, by Lemma 3.1, there exists \( \alpha \)-closed subsets \( H_{1/4} \) and \( H_{3/4} \) such that

\[
G_2 \subseteq H_{1/4} \subseteq \alpha(H_{1/4}^\Lambda) \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^\Lambda) \subseteq H_{3/4} \subseteq \alpha(H_{3/4}^\Lambda) \subseteq G_1^c.
\]

By continuing this method for every \( t \in D \), where \( D \subseteq [0, 1] \) is the set of rational numbers that their denominators are exponents of 2, we obtain \( \alpha \)-closed subsets \( H_t \) with the property that if \( t_1, t_2 \in D \) and \( t_1 < t_2 \), then \( H_{t_1} \subseteq H_{t_2} \). We define the function \( h \) on \( X \) by \( h(x) = \inf \{t : x \in H_t \} \) for \( x \notin G_1 \) and \( h(x) = 1 \) for \( x \in G_1 \).
Strong Insertion of a Contra-\(\alpha\)-continuous Function …

Note that for every \(x \in X\), \(0 \leq h(x) \leq 1\), i.e., \(h\) maps \(X\) into \([0, 1]\). Also, we note that for any \(t \in D\), \(G_2 \subseteq H_t\); hence \(h(G_2) = \{0\}\). Furthermore, by definition, \(h(G_1) = \{1\}\). It remains only to prove that \(h\) is a contra-\(\alpha\)-continuous function on \(X\). For every \(\alpha \in \mathbb{R}\), we have if \(\alpha \leq 0\), then \(\{x \in X : h(x) < \alpha\} = \emptyset\) and if \(0 < \alpha\), then \(\{x \in X : h(x) < \alpha\} = \bigcup[H_t : t < \alpha]\). hence, they are \(\alpha\)-closed subsets of \(X\). Similarly, if \(\alpha < 0\), then \(\{x \in X : h(x) > \alpha\} = X\) and if \(0 \leq \alpha\), then \(\{x \in X : h(x) > \alpha\} = \bigcup \{\alpha(H_i^\alpha)^c : t > \alpha\}\) hence, every of them is an \(\alpha\)-closed subset. Consequently \(h\) is a contra-\(\alpha\)-continuous function.

Lemma 3.3. Suppose that \(X\) is a topological space. If each pair of disjoint subsets \(G_1, G_2\) of \(X\), where \(G_1\) is preopen and \(G_2\) is semi-open, can separate by \(\alpha\)-closed subsets of \(X\), and \(G_1\) (resp. \(G_2\)) is an \(\alpha\)-closed subsets of \(X\), then there exists a contra-continuous function \(h : X \to [0, 1]\) such that, \(h^{-1}(0) = G_1\) (resp. \(h^{-1}(0) = G_2\)) and \(h(G_2) = \{1\}\) (resp. \(h(G_1) = \{1\}\)).

Proof. Suppose that \(G_1\) (resp. \(G_2\)) is an \(\alpha\)-closed subset of \(X\). By Lemma 3.2, there exists a contra-\(\alpha\)-continuous function \(h : X \to [0, 1]\) such that, \(h(G_1) = \{0\}\) (resp. \(h(G_2) = \{0\}\)) and \(h(X \setminus G_1) = \{1\}\) (resp. \(h(X \setminus G_2) = \{1\}\)). Hence, \(h^{-1}(0) = G_1\) (resp. \(h^{-1}(0) = G_2\)) and since \(G_2 \subseteq X \setminus G_1\) (resp. \(G_1 \subseteq X \setminus G_2\)), therefore \(h(G_2) = \{1\}\) (resp. \(h(G_1) = \{1\}\)).

Lemma 3.4. Suppose that \(X\) is a topological space such that every two disjoint semi-open and preopen subsets of \(X\) can be separated by \(\alpha\)-closed subsets of \(X\). The following conditions are equivalent:

(i) For every two disjoint subsets \(G_1\) and \(G_2\) of \(X\), where \(G_1\) is preopen and \(G_2\) is semi-open, there exists a contra-\(\alpha\)-continuous function \(h : X \to [0, 1]\) such that, \(h^{-1}(0) = G_1\) (resp. \(h^{-1}(0) = G_2\)) and \(h^{-1}(1) = G_2\) (resp. \(h^{-1}(1) = G_1\)).

(ii) Every preopen (resp. semi-open) subset of \(X\) is an \(\alpha\)-closed subsets of \(X\).

(iii) Every preclosed (resp. semi-closed) subset of \(X\) is an \(\alpha\)-open subsets of \(X\).
Proof. (i) \(\Rightarrow\) (ii) Suppose that \(G\) is a preopen (resp. semi-open) subset of \(X\). Since \(\emptyset\) is a semi-open (resp. preopen) subset of \(X\), by (i) there exists a contra-\(\alpha\)-continuous function \(h : X \to [0, 1]\) such that, \(h^{-1}(0) = G\). Set \(F_n = \{x \in X : h(x) < \frac{1}{n}\}\). Then for every \(n \in \mathbb{N}\), \(F_n\) is an \(\alpha\)-closed subset of \(X\) and \(\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G\).

(ii) \(\Rightarrow\) (i) Suppose that \(G_1\) and \(G_2\) are two disjoint subsets of \(X\), where \(G_1\) is preopen and \(G_2\) is semi-open. By Lemma 3.3, there exists a contra-\(\alpha\)-continuous function \(f : X \to [0, 1]\) such that, \(f^{-1}(0) = G_1\) and \(f(G_2) = \{1\}\). Set \(G = \{x \in X : f(x) < \frac{1}{2}\}\), \(F = \{x \in X : f(x) = \frac{1}{2}\}\), and \(H = \{x \in X : f(x) > \frac{1}{2}\}\). Then \(G \cup F\) and \(H \cup F\) are two \(\alpha\)-open subsets of \(X\) and \((G \cup F) \cap G_2 = \emptyset\). By Lemma 3.3, there exists a contra-\(\alpha\)-continuous function \(g : X \to \left[\frac{1}{2}, 1\right]\) such that, \(g^{-1}(1) = G_2\) and \(g(G \cup F) = \left\{\frac{1}{2}\right\}\). Define \(h\) by \(h(x) = f(x)\) for \(x \in G \cup F\), and \(h(x) = g(x)\) for \(x \in H \cup F\). Then \(h\) is well-defined and a contra-\(\alpha\)-continuous function, since \((G \cup F) \cap (H \cup F) = F\) and for every \(x \in F\) we have \(f(x) = g(x) = \frac{1}{2}\). Furthermore, \((G \cup F) \cup (H \cup F) = X\), hence \(h\) defined on \(X\) and maps to \([0, 1]\). Also, we have \(h^{-1}(0) = G_1\) and \(h^{-1}(1) = G_2\).

(ii) \(\Leftrightarrow\) (iii) By De Morgan law and noting that the complement of every \(\alpha\)-open subset of \(X\) is an \(\alpha\)-closed subset of \(X\) and complement of every \(\alpha\)-closed subset of \(X\) is an \(\alpha\)-open subset of \(X\), the equivalence is hold.

Corollary 3.5. If for every two disjoint subsets \(G_1\) and \(G_2\) of \(X\), where \(G_1\) is preopen (resp. semi-open) and \(G_2\) is semi-open (resp. preopen), there exists a contra-\(\alpha\)-continuous function \(h : X \to [0, 1]\) such that, \(h^{-1}(0) = G_1\) and \(h^{-1}(1) = G_2\), then \(X\) has the strong \(c\alpha\)-insertion property for \((cpc, csc)\) (resp. \((csc, cpc)\)).

Proof. Since for every two disjoint subsets \(G_1\) and \(G_2\) of \(X\), where \(G_1\) is preopen (resp. semi-open) and \(G_2\) is semi-open (resp. preopen), there exists a contra-\(\alpha\)-
Strong Insertion of a Contra-$\alpha$-continuous Function …

continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \left\{ x \in X : h(x) < \frac{1}{2} \right\}$ and $F_2 = \left\{ x \in X : h(x) > \frac{1}{2} \right\}$. Then $F_1$ and $F_2$ are two disjoint $\alpha$-closed subsets of $X$ that contain $G_1$ and $G_2$, respectively. Hence, by Corollary 3.4, $X$ has the weak $c\alpha$-insertion property for $(cpc, csc)$ and $(csc, cpc)$. Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ is $cpc$ (resp. $csc$) and $f$ is $cac$. Since $f - g$ is $cpc$ (resp. $csc$), therefore the lower cut set $A(f - g, 2^{-n}) = \{ x \in X : (f - g)(x) \leq 2^{-n} \}$ is a preopen (resp. semi-open) subset of $X$. Now setting $H_n = \{ x \in X : (f - g)(x) > 2^{-n} \}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, $H_n$ is an $\alpha$-open subset of $X$ and we have $\{ x \in X : (f - g)(x) > 0 \} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}$, $H_n$ and $A(f - g, 2^{-n})$ are disjoint subsets of $X$. By Lemma 3.2, $H_n$ and $A(f - g, 2^{-n})$ can be completely separated by contra-$\alpha$-continuous functions. Hence by Theorem 2.2, $X$ has the strong $c\alpha$-insertion property for $(cpc, cac)$ (resp. $(csc, c\alpha c)$).

By an analogous argument, we can prove that $X$ has the strong $c\alpha$-insertion property for $(cac, csc)$ (resp. $(c\alpha c, cpc)$). Hence, by Theorem 2.3, $X$ has the strong $c\alpha$-insertion property for $(cpc, csc)$ (resp. $(csc, cpc)$).

References


