The Convex \((\delta, L)\) Weak Contraction Mapping Theorem and its Non-Self Counterpart in Graphic Language

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Abstract

Let \((X, d)\) be a metric space. A map \(T : X \mapsto X\) is said to be a \((\delta, L)\) weak contraction [1] if there exists \(\delta \in (0, 1)\) and \(L \geq 0\) such that the following inequality holds for all \(x, y \in X\):

\[d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).\]

On the other hand, the idea of convex contractions appeared in [2] and [3]. In the first part of this paper, motivated by [1]-[3], we introduce a concept of convex \((\delta, L)\) weak contraction, and obtain a fixed point theorem associated with this mapping. In the second part of this paper, we consider the map is a non-self map, and obtain a best proximity point theorem. Finally, we leave the reader with some open problems.

1. Introduction and Preliminaries

The higher-order fixed point theory [4] is inspired by [5]. In particular, the idea of higher-order Banach mapping was defined as follows:

**Definition 1.1.** [5] Let \((X, d)\) be a metric space. A map \(T : X \mapsto X\) is called an \(r\)-th order Banach mapping if for all \(x, y \in X\), \(0 \leq c_q < 1\), \(0 \leq q \leq r - 1\), and \(r \in \mathbb{N}\), the following inequality holds
\[ d(T^r x, T^r y) \leq \sum_{q=0}^{r-1} c_q d(T^q x, T^q y) \]

with \( \sum_{q=0}^{r-1} c_q < 1 \).

**Remark 1.2.** A map \( T : X \mapsto X \) is called a convex contraction [2]-[3], if \( r = 2 \) in the definition immediately above.

By these observations we introduce the following

**Definition 1.3.** Let \( (X, d) \) be a metric space. A map \( T : X \mapsto X \) is called an \( r \)-th order \((\delta, L)\) weak contraction mapping if for all \( x, y \in X \), \( 0 < \delta_q < 1 \), \( L_q \geq 0 \), \( 0 \leq q \leq r - 1 \), and \( r \in \mathbb{N} \), the following inequality holds

\[ d(T^r x, T^r y) \leq \sum_{q=0}^{r-1} \{\delta_q d(T^q x, T^q y) + L_q d(T^q y, T^{q+1} x)\} \]

with \( \sum_{q=0}^{r-1} \delta_q < 1 \).

**Remark 1.4.** If \( r = 2 \) in the definition immediately above, then we say \( T : X \mapsto X \) is a convex \((\delta, L)\) weak contraction mapping. Note that if \( r = 1 \) in the above definition, then \( T : X \mapsto X \) is a \((\delta, L)\) weak contraction [1].

Also we recall the following results associated with the \((\delta, L)\) weak contraction

**Theorem 1.5.** [1] Let \( (X, d) \) be a complete metric space and \( T : X \mapsto X \) be an almost contraction, that is, a mapping for which there exist a constant \( \delta \in [0, 1) \) and some \( L \geq 0 \) such that for all \( x, y \in X \)

\[ d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx). \]

Then

(a) \( \text{Fix}(T) = \{ x \in X : Tx = x \} \neq \emptyset \).

(b) For any \( x_0 \in X \), the Picard iteration \( \{x_n\}_{n=0}^{\infty} \) given by \( x_{n+1} = Tx_n \), \( n = 1, 2, \ldots \) converges to some \( x^* \in \text{Fix}(T) \).
The following estimate holds
\[ d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_{n-1}, x_n) \]
for all \( n = 0, 1, 2, ...; i = 1, 2, ... \).

Theorem 1.6. [6] Let \( (X, d) \) be a complete metric space and \( T : X \rightarrow X \) be a weak contraction for which there exist a constant \( \theta \in (0, 1) \) and some \( L_1 \geq 0 \) such that
\[ d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx). \]
Then
(a) \( T \) has a unique fixed point, that is, \( F(T) = \{ x^* \} \).
(b) For any \( x_0 \in X \), the Picard iteration \( \{ x_n \}_{n=0}^\infty \) given by \( x_{n+1} = Tx_n \), \( n = 1, 2, ... \) converges to \( x^* \).
(c) The a priori and a posteriori error estimates holds
\[ d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1) \]
for \( n = 0, 1, 2, ... \); \[ d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n) \]
for \( n = 1, 2, ... \).
(d) The rate of convergence of the Picard iteration is given by
\[ d(x_n, x^*) \leq \theta d(x_{n-1}, x^*) \]
for \( n = 1, 2, ... \).

Now let \( W \) and \( V \) be two nonempty subsets of a metric space \( (X, d) \) and let \( S : W \rightarrow V \) be a non-self map. If \( W \cap V \) is nonempty, then the equation \( Sx = x \) may not have a solution. Naturally the following arises

Question 1.7. How far is the distance between \( x \) and \( Sx \)?
The problem of global optimization for determining the minimum value of the distance \( d(x, Sx) = \min\{d(x, y) : x \in W \text{ and } y \in V\} \) is the study of best proximity point theory. Since the early paper of [7], many best proximity point theorems have been obtained, and for example see references [9-23] contained in [8].

**Notation 1.8.** Throughout this paper

(a) \( W \) and \( V \) will denote nonempty subsets of a metric space \((X, d)\).

(b) \( d(W, V) := \inf\{d(x, y) : x \in W \text{ and } y \in V\}\).

(c) \( W_0 = \{x \in W : d(x, y) = d(W, V) \text{ for some } y \in V\}\).

(d) \( V_0 = \{y \in V : d(x, y) = d(W, V) \text{ for some } x \in W\}\).

The notion of proximal contraction appeared in [9], now we introduce the following

**Definition 1.9.** Let \( S : W \mapsto V \) be a non-self mapping. We say \( S \) is a proximal convex \((\delta, L)\) weak contraction if there exists \( \delta_0, \delta_1 \in (0, 1), \) \( L_0, L_1 \geq 0, \) and \( u_1, u_2, x, y \in W \) such that \( d(u_1, Sx) = d(W, V) \) and \( d(u_2, Sy) = d(W, V) \) implies
\[
d(Su_1, Su_2) \leq \delta_0 d(x, y) + L_0 d(y, u_1) + \delta_1 d(Sx, Sy) + L_1 d(Sy, Su_1).
\]

The notion of \( G \)-proximal Kannan mapping appeared in [8], now we introduce the following

**Definition 1.10.** Let \((X, d)\) be a metric space, and \( G = (V(G), E(G))\) be a directed graph such that \( V(G) = X\). A non-self mapping \( S : W \mapsto V \) is called a \( G \)-proximal convex \((\delta, L)\) weak contraction, if there exists \( \delta_0, \delta_1 \in (0, 1) \) and \( L_0, L_1 \geq 0 \), such that \((x, y) \in E(G), d(u, Sx) = d(W, V) \) and \( d(v, Sy) = d(W, V) \) implies
\[
d(Su, Sv) \leq \delta_0 d(x, y) + L_0 d(y, u) + \delta_1 d(Sx, Sy) + L_1 d(Sy, Su),
\]

where \( x, y, u, v \in W \).

**Definition 1.11.** [8] Let \((X, d)\) be a metric space and \( G = (V(G), E(G))\) be a directed graph such that \( V(G) = X\). A non-self mapping \( S : W \mapsto V \) is called proximally \( G \)-edge-preserving, if for each \( x, y, u, v \in W, (x, y) \in E(G), d(u, Sx) = d(W, V) \) and \( d(v, Sy) = d(W, V) \) implies \( (u, v) \in E(G) \).
The rest of this paper is organized as follows. In the next section we obtain a fixed point theorem associated with the convex \((\delta, L)\) weak contraction, and a best proximity point theorem for its non-self version endowed with a graph. We close this paper with some open problems suggested in Section 3.

2. Main Result

2.1. A fixed point theorem

**Theorem 2.1.** Let \((X, d)\) be a metric space, and \(T : X \mapsto X\) be a convex \((\delta, L)\) weak contraction mapping, that is, \(T\) satisfies

\[
d(T^2x, T^2y) \leq \delta_0d(x, y) + L_0d(y, Tx) + \delta_1d(Tx, Ty) + L_4d(Ty, T^2x)
\]

for all \(x, y \in X\) with \(0 < \delta_0, \delta_1 < 1,\ L_0, L_1 \geq 0,\) and \(\delta_0 + \delta_1 < 1.\) If \((X, d)\) is complete, then the fixed point of \(T\) exists. If, in addition, \(T\) is a convex \((\delta, L)\) weak contraction such that there exists \(0 < \delta_0, \delta_1 < 1,\ L_0^*, L_1^* \geq 0,\) with \(\delta_0 + \delta_1 < 1\) satisfying

\[
d(T^2x, T^2y) \leq \delta_0d(x, y) + L_0^*d(x, Tx) + \delta_1d(Tx, Ty) + L_4^*d(Tx, T^2x),
\]

then the fixed point is unique.

**Proof.** Define \(x_{n+1} = Tx_n = T^2x_{n-1}\) for all \(n \in \mathbb{N},\) and observe we have the following

\[
d(x_{n+1}, x_{n+2}) = d(T^2x_{n-1}, T^2x_n)
\]

\[
\leq \delta_0d(x_{n-1}, x_n) + L_0d(x_n, Tx_{n-1}) + \delta_1d(Tx_{n-1}, Tx_n)
\]

\[
+ L_4d(Tx_n, T^2x_{n-1})
\]

\[
= \delta_0d(x_{n-1}, x_n) + L_0d(x_n, x_n) + \delta_1d(x_n, x_{n+1}) + L_4d(x_{n+1}, x_{n+1})
\]

\[
= \delta_0d(x_{n-1}, x_n) + \delta_1d(x_n, x_{n+1})
\]

\[
\leq (\delta_0 + \delta_1) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}
\]

\[
= (\delta_0 + \delta_1) d(x_n, x_{n+1}).
\]
Set \( h := (\delta_0 + \delta_1) \), and observe by induction we have \( d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \) for all \( n \in \mathbb{N} \). For \( n, m \in \mathbb{N} \) with \( n < m \) we deduce the following

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\
\leq h^n d(x_0, x_1) + \cdots + h^{m-1} d(x_0, x_1) \\
\leq (h^n + h^{n+1} + \cdots) d(x_0, x_1) \\
= \frac{h^n}{1-h} d(x_0, x_1).
\]

Since \( h < 1 \), if we take limits in the above inequality as \( n, m \to \infty \) we deduce that \( \{x_n\} \) is Cauchy, and since \( X \) is complete, there is \( v \in X \) such that \( \lim_{n \to \infty} x_n = v \). Now we show the fixed point exist. Suppose \( v \) is a fixed point of \( T \) but not of \( T^2 \), then we know \( d(v, Tv) = 0 \), but \( d(v, T^2v) > 0 \). Now observe we have the following

\[
0 < d(v, T^2v) \\
\leq d(v, x_{n+1}) + d(x_{n+1}, T^2v) \\
= d(v, x_{n+1}) + d(T^2x_{n-1}, T^2v) \\
\leq d(v, x_{n+1}) + \delta_0 d(x_{n-1}, v) + L_0 d(v, Tx_{n-1}) + \delta_1 d(Tx_{n-1}, Tv) + L_1 d(Tv, T^2x_{n-1}) \\
= d(v, x_{n+1}) + \delta_0 d(x_{n-1}, v) + L_0 d(v, x_n) + \delta_1 d(x_n, Tv) + L_1 d(Tv, x_{n+1}).
\]

Taking limits in the above inequality and using the fact that \( d(v, Tv) = 0 \), we deduce that \( d(v, T^2v) \) is bounded above and below by zero, hence the assumption that \( d(v, T^2v) > 0 \) cannot be true, it must be the case that \( d(v, T^2v) = 0 \), that is, \( T^2v = v \).

It now follows that \( v \) is also a fixed point of \( T^2 \). Now we show the fixed point is unique. Suppose \( a = Ta = T^2a \) and \( b = Tb = T^2b \), but \( a \neq b \). From the second part of the theorem we deduce the following

\[
d(a, b) = d(T^2a, T^2b) \\
\leq \delta_0 d(a, b) + L_0^* d(a, Ta) + \delta_1 d(Ta, Tb) + L_1^* d(Ta, T^2a)
\]
\[ \delta_0 d(a, b) + \delta_1 d(a, b) \]

\[ \leq (\delta_0 + \delta_1) d(a, b). \]

Since \( 1 - (\delta_0 + \delta_1) \neq 0 \) and \( d > 0 \), from the above inequality we must have \( d(a, b) = 0 \) and hence \( a = b \), which contradicts the assumption that \( a \neq b \). Thus, the fixed point is unique.

2.2. A best proximity point theorem

Theorem 2.2. Let \((X, d)\) be a complete metric space, \(G = (V(G), E(G))\) be a directed graph such that \(V(G) = X\). Let \(W\) and \(V\) be nonempty closed subsets of \(X\) with \(W_0\) nonempty. Let \(S : W \mapsto V\) be a non-self mapping satisfying the following properties:

(a) \(S\) is proximally \(G\)-edge-preserving, continuous and \(G\)-proximal convex \((\delta, L)\) weak contraction such that \(S(W_0) \subseteq V_0\).

(b) there exist \(x_0, x_1 \in W_0\) such that

\[ d(x_1, Sx_0) = d(W, V), \ d(x_2, Sx_1) = d(x_2, S^2x_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G). \]

Then \(S\) has a best proximity point in \(W\), that is, there exists an element \(w \in W\) such that \(d(w, Sw) = d(W, V)\) and \(d(w, S^2w) = d(W, V)\). Further the sequence \(\{x_n\}\) defined by

\[ d(x_n, Sx_{n-1}) = d(W, V) \text{ and } d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2x_{n-1}) = d(W, V) \]

for all \(n \in \mathbb{N}\) converges to the element \(w\).

Proof. From condition (b), there exist \(x_0, x_1 \in W_0\) such that

\[ d(x_1, Sx_0) = d(W, V), \ d(x_2, Sx_1) = d(x_2, S^2x_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G). \]

(1)

Since \(S(W_0) \subseteq V_0\), we have \(Sx_0 \in V_0\) and hence there exists \(x_3 \in W_0\) such that

\[ d(x_3, Sx_2) = d(W, V) \text{ and } d(x_4, Sx_3) = d(x_4, S^2x_2) = d(W, V). \]

(2)

By the proximally \(G\)-edge preserving of \(S\) and using both (1) and (2), we get

\( (x_3, x_4), (x_2, x_3) \in E(G) \).
By continuing this process, we can form the sequence \( \{x_n\} \) in \( W_0 \) such that
\[
d(x_n, Sx_{n-1}) = d(W, V) \quad \text{and} \quad d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2x_{n-1}) = d(W, V)
\]
with \( (x_{n-1}, x_n) \in E(G) \), for all \( n \in \mathbb{N} \). \( \text{(3)} \)

Next we show that \( S \) has a best proximity point in \( W \). Suppose there exists \( n_0 \in \mathbb{N} \) such that \( x_{n_0} = x_{n_0+1} \). By using (3), we obtain that
\[
d(x_{n_0}, Sx_{n_0}) = d(x_{n_0+1}, Sx_{n_0}) = d(W, V)
\]
and
\[
d(x_{n_0}, S^2x_{n_0}) = d(x_{n_0+1}, S^2x_{n_0}) = d(x_{n_0+2}, S^2x_{n_0}) = d(W, V)
\]
and so \( x_{n_0} \) is a best proximity point of \( S \) and of \( S^2 \). Now we suppose that \( x_{n-1} \neq x_n \) for all \( n \in \mathbb{N} \). We show that \( \{x_n\} \) is a Cauchy sequence in \( W \). As \( S \) is \( G \)-proximal convex \( (\delta, L) \) weak contraction, and for each \( n \in \mathbb{N} \), \( (x_{n-1}, x_n) \in E(G) \), \( d(x_n, Sx_{n-1}) = d(W, V) \) and \( d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2x_{n-1}) = d(W, V) \), then we have
\[
d(x_{n+1}, x_{n+2}) \leq \delta_0 d(x_{n-1}, x_n) + L_0 d(x_n, x_{n+1}) + \delta_1 d(x_n, x_{n+1}) + L_1 d(x_{n+1}, x_{n+1})
\]
\[
= \delta_0 d(x_{n-1}, x_n) + \delta_1 d(x_n, x_{n+1})
\]
\[
\leq (\delta_0 + \delta_1) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}
\]
\[
= (\delta_0 + \delta_1) d(x_n, x_{n+1}).
\]
Now set \( h := \delta_0 + \delta_1 \). By the above inequality we have
\[
d(x_1, x_2) \leq h d(x_0, x_1)
\]
and hence
\[
d(x_2, x_3) \leq h^2 d(x_0, x_1).
\]
By induction, we deduce the following
\[
d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)
\] \( \text{(4)} \)
for all \( n \in \mathbb{N} \cup \{0\} \). From (4), for each \( m, n \in \mathbb{N} \) with \( m > n \), we deduce the following
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\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)
\]

\[
\leq h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \cdots + h^{m-1} d(x_0, x_1)
\]

\[
= d(x_0, x_1) \sum_{i=n}^{m-1} h^i
\]

\[
= \frac{h^n}{1-h} d(x_0, x_1).
\]

Since \(h \in (0, 1)\), it follows that \(\{x_n\}\) is a Cauchy sequence in \(W\). Since \(W\) is closed, there exists \(w \in W\) such that \(x_n \to w\). By continuity of \(S\) and of \(S^2\), we have \(Sx_n \to Sw\) and \(S^2x_n \to S^2w\) as \(n \to \infty\). As the metric function is continuous, we obtain

\[
d(x_{n+1}, Sx_n) \to d(w, Sw) \text{ as } n \to \infty
\]

and

\[
d(x_{n+2}, Sx_{n+1}) = d(x_{n+2}, S^2x_n) \to d(w, S^2w) \text{ as } n \to \infty.
\]

Similarly, by (3), we have

\[
d(w, Sw) = d(W, V) \text{ and } d(w, S^2w) = d(W, V).
\]

It follows that \(w \in W\) is a best proximity point of \(S\) and of \(S^2\). Moreover, the sequence \(\{x_n\}\) defined by

\[
d(x_{n+1}, Sx_n) = d(W, V) \text{ and } d(x_{n+2}, Sx_{n+1}) = d(x_{n+2}, S^2x_n) = d(W, V),
\]

\[
n \in \mathbb{N} \cup \{0\}
\]

converges to an element \(w\), and the proof is completed. \(\square\)

3. Open Problem

Definition 3.1. [10] Let \((X, d)\) be a metric space. A map \(T : X \mapsto X\) is called a Reich contraction if there exist nonnegative constants \(a, b, c\) with \(a + b + c < 1\) such that the following holds for all \(x, y \in X\)
\[ d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty). \]

Note if \( T \) is a Reich contraction, then it also satisfies the following
\[ d(Tx, Ty) \leq (a + b + c) \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \]

Therefore we have the following

**Definition 3.2.** [11] Let \((X, d)\) be a metric space. A map \( T : X \mapsto X \) is also a Reich contraction if there exists \( k \in \left[0, \frac{1}{3}\right) \) such that the following holds for all \( x, y \in X \)
\[ d(Tx, Ty) \leq k[d(x, y) + d(x, Tx) + d(y, Ty)]. \]

Related to the Reich contraction, the following was obtained

**Theorem 3.3.** [10] Let \((X, d)\) be a metric space, and \( T : X \mapsto X \) be a Reich contraction. Then \( T \) has a unique fixed point, provided \((X, d)\) is complete.

Now we introduced the following, as the Berinde characterization of the Reich contraction

**Definition 3.4.** [11] Let \((X, d)\) be a metric space. A map \( T : X \mapsto X \) is called a Berinde weak Reich contraction, if there exists \( \delta \in \left[0, \frac{1}{3}\right) \) and \( L \geq 0 \) such that the following holds for all \( x, y \in X \)
\[ d(Tx, Ty) \leq \delta[d(x, y) + d(x, Tx) + d(y, Ty)] + Ld(y, Tx). \]

Note that if \( a = Ta \) and \( b = Tb \), but \( a \neq b \), then we have the following inequality from Definition 3.4
\[ d(a, b) = d(Ta, Tb) \]
\[ \leq \delta[d(a, b) + d(a, Ta) + d(b, Tb)] + Ld(b, Ta) \]
\[ = \delta[d(a, b) + d(a, a) + d(b, b)] + Ld(b, a) \]
\[ = \delta d(a, b) + Ld(b, a) \]
\[ = (\delta + L)d(a, b). \]
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Unless $1 - (\delta + L) > 0$ we cannot conclude $a = b$. So we introduced the following as a way to "force" uniqueness of the fixed point.

**Definition 3.5.** [11] Let $(X, d)$ be a metric space. A map $T : X \mapsto X$ is called a $(\delta, 1 - 3\delta)$ weak Reich contraction if the following holds for all $x, y \in X$ and $\delta \in \left(0, \frac{1}{3}\right]$

$$d(Tx, Ty) \leq \delta [d(x, y) + d(x, Tx) + d(y, Ty)] + (1 - 3\delta)d(y, Tx).$$

Note that if $L = 0$ in Definition 3.4, then we recover Definition 3.2. Note that for any $\delta \in \left(0, \frac{1}{3}\right]$, Definition 3.5 does not reduce to Definition 3.2.

The following result is previously known

**Theorem 3.6.** [11] Let $(X, d)$ be a metric space, and $T : X \mapsto X$ be a $(\delta, 1 - 3\delta)$ weak Reich contraction. Then $T$ has a unique fixed point provided $X$ is complete.

Our first open problem introduces a so-called convex $\delta, L$ weak Reich contraction mapping theorem

**Conjecture 3.7.** Let $(X, d)$ be a metric space, and $T : X \mapsto X$ be a convex $\delta, L$ weak Reich contraction mapping, that is, $T$ satisfies

$$d(T^2x, T^2y) \leq \delta_0 [d(x, y) + d(x, Tx) + d(y, Ty)] + L_0d(y, Tx)$$

$$+ \delta_1 [d(Tx, Ty) + d(Tx, T^2x) + d(Ty, T^2y)] + L_1d(Ty, T^2x)$$

for all $x, y \in X$ with $0 < \delta_0, \delta_1 < \frac{1}{3}$, $L_0, L_1 \geq 0$, and $\delta_0 + \delta_1 < 1$. If $(X, d)$ is complete, then the fixed point of $T$ exists. If in addition, $T$ is a convex $\delta, L$ weak Reich contraction such there exists $0 < \delta_0, \delta_1 < \frac{1}{3}$, $L_0^*, L_1^* \geq 0$, with $\delta_0 + \delta_1 < 1$ satisfying

$$d(T^2x, T^2y) \leq \delta_0 [d(x, y) + d(x, Tx) + d(y, Ty)] + L_0^*d(x, Tx)$$

$$+ \delta_1 [d(Tx, Ty) + d(Tx, T^2x) + d(Ty, T^2y)] + L_1^*d(Tx, T^2x),$$

then the fixed point is unique.
Sequel to the second open problem, we will need the following

**Definition 3.8.** Let $S : W \mapsto V$ be a non-self mapping. We say $S$ is a proximal convex $(\delta, L)$ weak Reich contraction if there exist $\delta_0, \delta_1 \in \left[0, \frac{1}{3}\right]$, $L_0, L_1 \geq 0$, and $u_1, u_2, x, y \in W$ such that $d(u_1, Sx) = d(W, V)$ and $d(u_2, Sy) = d(W, V)$ implies

$$d(Su_1, Su_2) \leq \delta_0[d(x, y) + d(x, u_1) + d(y, u_2)] + L_0d(y, u_1) + \delta_1[d(Sx, Sy) + d(Sx, Su_1) + d(Sy, Su_2)] + L_1d(Sy, Su_1).$$

**Definition 3.9.** Let $(X, d)$ be a metric space, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called a $G$-proximal convex $(\delta, L)$ weak Reich contraction, if there exists $\delta_0, \delta_1 \in \left[0, \frac{1}{3}\right]$ and $L_0, L_1 \geq 0$ such that $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$, and $d(v, Sy) = d(W, V)$ implies

$$d(Su, Sv) \leq \delta_0[d(x, y) + d(x, u) + d(y, v)] + L_0d(y, u) + \delta_1[d(Sx, Sy) + d(Sx, Su) + d(Sy, Sv)] + L_1d(Sy, Su),$$

where $x, y, u, v \in W$.

Now we have the following which can be regarded as the non-self counterpart to Conjecture 3.7 in graphic language

**Conjecture 3.10.** Let $(X, d)$ be a complete metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Let $W$ and $V$ be nonempty closed subsets of $X$ with $W_0$ nonempty. Let $S : W \mapsto V$ be a non-self mapping satisfying the following properties:

(a) $S$ is proximally $G$-edge-preserving, continuous and $G$-proximal convex $(\delta, L)$ weak Reich contraction such that $S(W_0) \subset V_0$

(b) there exist $x_0, x_1 \in W_0$ such that

$$d(x_1, Sx_0) = d(W, V), \ d(x_2, Sx_1) = d(x_2, S^2x_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Then $S$ has a best proximity point in $W$, that is, there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$ and $d(w, S^2w) = d(W, V)$. Further the sequence $\{x_n\}$ defined
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by

$$d(x_n, Sx_{n-1}) = d(W, V) \text{ and } d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2x_{n-1}) = d(W, V)$$

for all $n \in \mathbb{N}$ converges to the element $w$.

References


