Some $KV$ Indices of Certain Dendrimers

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Abstract

In this paper, we define the modified first and second $KV$ indices, $F$-$KV$ and $F_1$-$KV$ indices, hyper $F$-$KV$ index and augmented $KV$ index of a graph and compute exact formulas for POPAM and tetrathiafulvalene dendrimers. Furthermore, we determine the $F$-$KV$, hyper $F$-$KV$ and augmented $KV$ polynomials of POPAM dendrimers and tetrathiafulvalene dendrimers.

1. Introduction

A molecular graph is a simple graph such that its vertices correspond to the atoms and edges to the bonds. Chemical Graph Theory is a branch of Mathematical Chemistry which has an important effect on the development of Chemical Sciences. Numerous topological indices have been considered in Theoretical Chemistry, especially in QSAR/QSPR study, see [1, 2].

Let $V(G)$, $E(G)$ be a vertex set and an edge set of a finite simple connected graph $G$ respectively. The degree $d(v)$ of a vertex $v$ is the number of edges incident to $v$. Let $M_G(v)$ denote the product of the degrees of all vertices adjacent to a vertex $v$. We refer to [3] for undefined term and notation.

In [4], Kulli introduced the first and second $KV$ indices, defined as
We introduce the modified first and second KV indices of a graph, defined as

\[ KV_1(G) = \sum_{uv \in E(G)} [M_G(u) + M_G(v)], \quad KV_2(G) = \sum_{uv \in E(G)} M_G(u)M_G(v). \]

In [5], Furtula and Gutman proposed the F-index of a graph \( G \), defined as

\[ F(G) = \sum_{u \in V(G)} d(u)^3 = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2]. \]

The F-index was studied, for example, in [6, 7, 8, 9, 10, 11].

We introduce the \( F_1-KV \) index of a graph \( G \), defined as

\[ F_1KV(G) = \sum_{uv \in E(G)} [M_G(u)^2 + M_G(v)^2]. \]

We define the \( F_1-KV \) polynomial of a graph \( G \) as

\[ F_1KV(G, x) = \sum_{uv \in E(G)} x^{[M_G(u)^2 + M_G(v)^2]}. \]

We define the harmonic KV index of a graph \( G \) as

\[ HKV(G) = \sum_{uv \in E(G)} \frac{2}{M_G(u) + M_G(v)}. \]

We propose the general harmonic KV index of a graph \( G \) and it is defined as

\[ HKV^a(G) = \sum_{uv \in E(G)} \left( \frac{2}{M_G(u) + M_G(v)} \right)^a. \]

The harmonic index was studied in [12, 13, 14].
The augmented $KV$ index of a graph $G$ is defined as

$$\text{AKVI}(G) = \sum_{uv \in E(G)} \left( \frac{M_G(u)M_G(v)}{M_G(u) + M_G(v) - 2} \right)^3.$$ \hspace{1cm} (6)

The augmented index was studied in [15, 16, 17].

Considering the augmented $KV$ index, we introduce the augmented $KV$ polynomial of a graph $G$ as

$$\text{AKVI}(G, x) = \sum_{uv \in E(G)} x \left( \frac{M_G(u)M_G(v)}{M_G(u) + M_G(v) - 2} \right)^3.$$ \hspace{1cm} (7)

We propose the hyper $F$-$KV$ index and hyper $F$-$KV$ polynomial of a graph as follows:

The hyper $F$-$KV$ index of a graph $G$ is defined as

$$\text{HFKV}(G) = \sum_{uv \in E(G)} \left[ M_G(u)^2 + M_G(v)^2 \right]^2.$$ \hspace{1cm} (8)

The hyper $F$-$KV$ polynomial of a graph $G$ is defined as

$$\text{HFKV}(G, x) = \sum_{uv \in E(G)} x \left[ M_G(u)^2 + M_G(v)^2 \right]^2.$$ \hspace{1cm} (9)

Very recently, some new $KV$ indices have been introduced and studied such as hyper $KV$ and square $KV$ indices [13], connectivity $KV$ indices [19], multiplicative connectivity $KV$ indices [20], multiplicative $KV$ indices and multiplicative hyper $KV$ indices [21]. In this paper, we compute the modified first and second $KV$ indices, $F$-$KV$ and hyper $F$-$KV$ indices, general harmonic $KV$ index, augmented $KV$ index of POPAM and tetrathiafulvalene dendrimers. Also the $F$-$KV$ polynomial, $F_1$-$KV$ polynomial, augmented $KV$ polynomial of POPAM and tetrathiafulvalene dendrimers are determined. For dendrimers see [22].

2. Results for POPAM Dendrimers

The family of POPAM dendrimers is symbolized by $POD_2[n]$, where $n$ is the steps of growth in this type of dendrimers. The graph of $POD_2[2]$ is shown in Figure 1.
Figure 1. The graph of $POD_2[2]$. 

Let $G$ be the graph of a POPAM dendrimer $POD_2[n]$. By algebraic method, we obtain that $G$ has $2^{n+5} - 10$ vertices and $2^{n+5} - 11$ edges. The edge partition of $POD_2[n]$ based on the degree product of neighbors of end vertices of each edge is given in Table 1.

<table>
<thead>
<tr>
<th>$M_G(u), M_G(v)uv \in E(G)$</th>
<th>(2, 2)</th>
<th>(2, 4)</th>
<th>(4, 4)</th>
<th>(4, 6)</th>
<th>(6, 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges</td>
<td>$2^{n+2}$</td>
<td>$2^{n+2}$</td>
<td>1</td>
<td>$3 \times 2^{n+2} - 6$</td>
<td>$3 \times 2^{n+2} - 6$</td>
</tr>
</tbody>
</table>

**Theorem 1.** The modified first and second KV indices of a POPAM dendrimer $POD_2[n]$ are given by

(i) $mKV_1(POD_2[n]) = \frac{391}{420} 2^{n+2} - \frac{253}{280},$

(ii) $mKV_2(POD_2[n]) = \frac{9}{16} 2^{n+2} - \frac{5}{16}.$

**Proof.** Let $G$ be the graph of $POD_2[n]$.

(i) From equation (1) and by using Table 1, we deduce
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\[ m_{KV_1}(POD_2[n]) = \sum_{u\in E(G)} \frac{1}{M_G(u) + M_G(v)} \]
\[ = \left( \frac{1}{2 + 2} \right) 2^{n+2} + \left( \frac{1}{2 + 4} \right) 2^{n+2} + \left( \frac{1}{4 + 4} \right) 3 \times 2^{n+2} - 6 \]
\[ + \left( \frac{1}{6 + 8} \right)(3 \times 2^{n+2} - 6) \]
\[ = \frac{391}{420} 2^{n+2} - \frac{253}{280} \]

(ii) By using equation (2) and Table 2, we obtain

\[ m_{KV_2}(POD_2[n]) = \sum_{u\in E(G)} \frac{1}{M_G(u)M_G(v)} \]
\[ = \left( \frac{1}{2 \times 2} \right) 2^{n+2} + \left( \frac{1}{2 \times 4} \right) 2^{n+2} + \left( \frac{1}{4 \times 4} \right) 3 \times 2^{n+2} - 6 \]
\[ + \left( \frac{1}{6 \times 8} \right)(3 \times 2^{n+2} - 6) \]
\[ = \frac{9}{16} 2^{n+2} - \frac{5}{16} \]

**Theorem 2.** The \( F_1 \)-KV index and its polynomial of a POPAM dendrimer \( POD_2[n] \) are given by

(i) \( F_1KV(POD_2[n]) = 484 \times 2^{n+2} - 880. \)

(ii) \( F_1KV(POD_2[n], x) = 2^{n+2} x^8 + 2^{n+2} x^{20} + x^{32} + (3 \times 2^{n+2} - 6) x^{52} + (3 \times 2^{n+2} - 6) x^{100}. \)

**Proof.** Let \( G \) be the graph of a POPAM dendrimer \( POD_2[n] \).

(i) From equation (3) and using Table 1, we derive

\[ F_1KV(POD_2[n]) = \sum_{u\in E(G)} [M_G(u)^2 + M_G(v)^2] \]
\[= (2^2 + 2^2)2^{n+2} + (2^2 + 4^2)2^{n+2} + (4^2 + 4^2)\]
\[+ (4^2 + 6^2)(3 \times 2^{n+2} - 6) + (6^2 + 8^2)(3 \times 2^{n+2} - 6)\]
\[= 484 \times 2^{n+2} - 880.\]

(ii) By using equation (4) and Table 1, we have

\[F_1KV(POD_2[n], x) = \sum_{u \neq v \in E(G)} [M_G(u)^2 + M_G(v)^2] \]
\[= 2^{n+2}x^{2^2+2^2} + 2^{n+2}x^{2^2+4^2} + x^{4^2+4^2}\]
\[+ (3 \times 2^{n+2} - 6)x^2 + 6^2 + (3 \times 2^{n+2} - 6)x^6 + 8^2\]
\[= 2^{n+2}x^8 + 2^{n+2}x^{20} + x^{32}\]
\[+ (3 \times 2^{n+2} - 6)x^{52} + (3 \times 2^{n+2} - 6)x^{100}.\]

**Theorem 3.** The general harmonic KV index of \(POD_2[n]\) is

\[HKV^a(POD_2[n]) = \left[\left(\frac{1}{2}\right)^a + \left(\frac{1}{3}\right)^a\right]2^{n+2} + \left[\left(\frac{1}{5}\right)^a + \left(\frac{1}{7}\right)^a\right] (3 \times 2^{n+2} - 6) + \left(\frac{1}{4}\right)^a. \ (10)\]

**Proof.** Let \(G = POD_2[n]\). By using equation (5) and Table 1, we deduce

\[HKV^a(POD_2[n]) = \sum_{u \neq v \in E(G)} \left(\frac{2}{M_G(u) + M_G(v)}\right)^a \]
\[= \left(\frac{2}{2+2}\right)^a 2^{n+2} + \left(\frac{2}{2+4}\right)^a 2^{n+2} + \left(\frac{2}{4+4}\right)^a \]
\[+ \left(\frac{2}{4+6}\right)^a (3 \times 2^{n+2} - 6) + \left(\frac{2}{6+8}\right)^a (3 \times 2^{n+2} - 6)\]
\[= \left[\left(\frac{1}{2}\right)^a + \left(\frac{1}{3}\right)^a\right]2^{n+2} + \left[\left(\frac{1}{5}\right)^a + \left(\frac{1}{7}\right)^a\right] (3 \times 2^{n+2} - 6) + \left(\frac{1}{4}\right)^a.\]
Corollary 3.1. The harmonic KV index of $POD_2[n]$ is
\[
\frac{391}{210} 2^{n+2} - \frac{253}{140}.
\]

Proof. Put $a = 1$ in equation (10), we get the desired result.

Theorem 4. The augmented KV index and its polynomial of a POPAM dendrimer $POD_2[n]$ are given by

(i) $AKVI(POD_2[n]) = 289 \times 2^{n+2} - \frac{14230}{27}$.

(ii) $AKVI(POD_2[n], x) = 2 \times 2^{n+2} x^8 + (3 \times 2^{n+2} - 6) x^{27}$
\[
+ (3 \times 2^{n+2} - 6) x^{64} + x^{27}.
\]

Proof. Let $G = POD_2[n]$.

(i) From equation (6) and by using Table 1, we deduce
\[
AKVI(POD_2[n]) = \sum_{uv \in E(G)} \left( \frac{M_G(u)M_G(v)}{M_G(u) + M_G(v) - 2} \right)^3
\]
\[
= \left( \frac{2 \times 2}{2 + 2 - 2} \right)^3 2^{n+2} + \left( \frac{2 \times 4}{2 + 4 - 2} \right)^3 2^{n+2} + \left( \frac{4 \times 4}{4 + 4 - 2} \right)^3
\]
\[
+ \left( \frac{4 \times 6}{4 + 6 - 2} \right)^3 (3 \times 2^{n+2} - 6) + \left( \frac{6 \times 8}{6 + 8 - 2} \right)^3 (3 \times 2^{n+2} - 6)
\]
\[
= 289 \times 2^{n+2} - \frac{14230}{27}.
\]

(ii) By using equation (7) and Table 1, we derive
\[
AKVI(POD_2[n], x) = \sum_{uv \in E(G)} x^\left( \frac{M_G(u)M_G(v)}{M_G(u) + M_G(v) - 2} \right)^3
\]
\[
= 2^{n+2} x^\left( \frac{2 \times 2}{2 + 2 - 2} \right)^3 + 2^{n+2} x^\left( \frac{2 \times 4}{2 + 4 - 2} \right)^3 + x^\left( \frac{4 \times 4}{4 + 4 - 2} \right)^3
\]
\[ + (3 \times 2^{n+2} - 6)x^6 \left( \frac{4 \times 6}{4+6-2} \right)^3 + (3 \times 2^{n+2} - 6)x^\left( \frac{6 \times 8}{6+8-2} \right)^3 \]

\[ = 2 \times 2^{n+2}x^8 + (3 \times 2^{n+2} - 6)x^{27} + (3 \times 2^{n+2} - 6)x^{64} + \frac{512}{x^{27}}. \]

Theorem 5. The hyper F-KV index and its polynomial of a POPAM dendrimer \( POD_2[n] \) are given by

(i) \( HFKV(POD_2[n]) = 38576 \times 2^{n+2} - 75200. \)

(ii) \( HFKV(POD_2[n], x) = 2^{n+2}x^{64} + 2^{n+2}x^{400} + x^{1024} \)

\[ + (3 \times 2^{n+2} - 6)x^{8112} + (3 \times 2^{n+2} - 6)x^{30000}. \]

Proof. Let \( G = POD_2[n] \).

(i) From equation (8) and using Table 1, we deduce

\[ HFKV(POD_2[n]) = \sum_{u \in E(G)} [M_G(u)^2 + M_G(v)^2]^2 \]

\[ = (2^2 + 2^2)^2 2^{n+2} + (2^2 + 4^2)^2 2^{n+2} + (4^2 + 4^2)^2 \]

\[ + (4^2 + 6^2)^2 (3 \times 2^{n+2} - 6) + (6^2 + 8^2)^2 (3 \times 2^{n+2} - 6) \]

\[ = 38576 \times 2^{n+2} - 75200. \]

(ii) From equation (9) and Table 1, we obtain

\[ HFKV(POD_2[n], x) = \sum_{u \in E(G)} [M_G(u)^2 + M_G(v)^2]^2 x \]

\[ = 2^{n+2}x^{(2^2+2^2)^2} + 2^{n+2}x^{(2^2+4^2)^2} + x^{(4^2+4^2)^2} \]

\[ + (3 \times 2^{n+2} - 6)x^{(4^2+6^2)^2} + (3 \times 2^{n+2} - 6)x^{(6^2+8^2)^2} \]

\[ = 2^{n+2}x^{64} + 2^{n+2}x^{400} + x^{1024} + (3 \times 2^{n+2} - 6)x^{8112} \]

\[ + (3 \times 2^{n+2} - 6)x^{30000}. \]

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3. Results for Tetrathiafulvalene Dendrimers

We consider the family of tetrathiafulvalene dendrimers. This family of dendrimers is symbolized by $TD_2[n]$, where $n$ is the steps of growth in this type of dendrimers. The graph of $TD_2[2]$ is presented in Figure 2.

Figure 2. The graph of $TD_2[2]$.

Let $G$ be the graph of a tetrathiafulvalene dendrimer $TD_2[n]$. By calculation, we obtain that $G$ has $31 \times 2^{n+2} - 24$ vertices and $32 \times 2^{n+2} - 85$ edges. The edge partition of $G$ based on the degree product of neighbors of end vertices of each edge is given in Table 2.

Table 2. Edge partition of $TD_2[n]$.

<table>
<thead>
<tr>
<th>$M_G(u), M_G(v)uv \in E(G)$</th>
<th>Number of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 3)</td>
<td>$2^{n+2}$</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>$2^{n+2} - 4$</td>
</tr>
<tr>
<td>(3, 8)</td>
<td>$2^{n+2}$</td>
</tr>
<tr>
<td>(6, 6)</td>
<td>$7 \times 2^{n+2} - 16$</td>
</tr>
</tbody>
</table>
Theorem 6. The modified first and second KV indices of $TD_2[n]$ are given by

(i) $mKV_1(TD_2[n]) = \frac{562}{165}2^{n+2} - \frac{13997}{2520}$.

(ii) $mKV_2(TD_2[n]) = \frac{53}{72}2^{n+2} - \frac{85}{54}$.

Proof. Let $G = TD_2[n]$,

(i) By using equation (1) and Table 2, we obtain

$$mKV_1(TD_2[n]) = \frac{1}{\sum_{u \in E(G)} M_G(u) + M_G(v)}$$

$$= \left(\frac{1}{2+3}\right)2^{n+2} + \left(\frac{1}{3+6}\right)(2^{n+2} - 4) + \left(\frac{1}{3+8}\right)2^{n+2}$$

$$+ \left(\frac{1}{6+6}\right)(2^{n+2} - 4) + \left(\frac{1}{6+8}\right)(7 \times 2^{n+2} - 16) + \left(\frac{1}{6+12}\right)(11 \times 2^{n+2} - 24)$$

$$+ \left(\frac{1}{6+9}\right)(2^{n+2} - 4) + \left(\frac{1}{6+12}\right)(3 \times 2^{n+2} - 8)$$

$$+ \left(\frac{1}{9+12}\right)(8 \times 2^{n+2} - 24) + \left(\frac{1}{12+12}\right)(2 \times 2^{n+2} - 5)$$

$$= \frac{562}{165}2^{n+2} - \frac{13997}{2520}.$$

(ii) By using equation (2) and Table 2, we deduce
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Theorem 7. The $F_{KV}$ index and its polynomial of $TD_2[n]$ are given by

(i) $F_{KV}(TD_2[n]) = 4768 \times 2^{n+2} - 12480$.

(ii) $F_{KV}(TD_2[n], x) = 2^{n+2}x^{13} + (2^{n+2} - 4)x^{45} + 2^{n+2}x^{73}$

$+ (7 \times 2^{n+2} - 16)x^{72} + (11 \times 2^{n+2} - 24)x^{225}$

$+ (2^{n+2} - 4)x^{117} + (3 \times 2^{n+2} - 8)x^{180}$

$+ (8 \times 2^{n+2} - 24)x^{225} + (2 \times 2^{n+2} - 5)x^{288}$.

Proof. Let $G = TD_2[n]$.

(i) From equation (3) and using Table 2, we derive

\[ mKV_2(TD_2[n]) = \sum_{u \neq v \in E(G)} \frac{1}{M_G(u)M_G(v)} \]

\[ = \left(\frac{1}{2 \times 3}\right)2^{n+2} + \left(\frac{1}{3 \times 6}\right)(2^{n+2} - 4) + \left(\frac{1}{3 \times 8}\right)2^{n+2} \]

\[ + \left(\frac{1}{6 \times 6}\right)(7 \times 2^{n+2} - 16) + \left(\frac{1}{6 \times 8}\right)(11 \times 2^{n+2} - 24) \]

\[ + \left(\frac{1}{6 \times 9}\right)(2^{n+2} - 4) + \left(\frac{1}{6 \times 12}\right)(3 \times 2^{n+2} - 8) \]

\[ + \left(\frac{1}{9 \times 12}\right)(8 \times 2^{n+2} - 24) + \left(\frac{1}{12 \times 12}\right)(2 \times 2^{n+2} - 5) \]

\[ = \frac{53}{72} \times 2^{n+2} - \frac{85}{54}. \]
\[ + (6^2 + 9^2)(2^{n+2} - 4) + (6^2 + 12^2)(3 \times 2^{n+2} - 8) + (9^2 + 12^2)(8 \times 2^{n+2} - 24) + (12^2 + 12^2)(2 \times 2^{n+2} - 5) = 4768 \times 2^{n+2} - 12480. \]

(ii) From equation (4) and using Table 2, we deduce

\[
F_1KV(TD_2[n], x) = \sum_{uv \in E(G)} x^{[M_G(u)^2 + M_G(v)^2]}
\]

\[
= 2^{n+2}x(2^2+3^2) + (2^{n+2} - 4)x(3^2+6^2) + 2^{n+2}x(3^2+8^2) + (7 \times 2^{n+2} - 16)x(4^2+6^2) + (11 \times 2^{n+2} - 24)x(6^2+8^2) + (2^{n+2} - 4)x(6^2+9^2) + (3 \times 2^{n+2} - 8)x(6^2+12^2) + (8 \times 2^{n+2} - 24)x(9^2+12^2) + (2 \times 2^{n+2} - 5)x(12^2+12^2) = 2^{n+2}x^{13} + (2^{n+2} - 4)x^{45} + 2^{n+2}x^{73} + (7 \times 2^{n+2} - 16)x^{72} + (11 \times 2^{n+2} - 24)x^{225} + (2^{n+2} - 4)x^{117} + (3 \times 2^{n+2} - 8)x^{180} + (8 \times 2^{n+2} - 24)x^{225} + (2 \times 2^{n+2} - 5)x^{288}.
\]

**Theorem 8.** The general harmonic KV index of \( TD_2[n] \) is given by

\[
HKV^a(TD_2[n]) = \left[ \left( \frac{2}{5} \right)^a + \left( \frac{2}{9} \right)^a + \left( \frac{2}{11} \right)^a + 7 \left( \frac{1}{6} \right)^a + 11 \left( \frac{1}{7} \right)^a + \left( \frac{2}{15} \right)^a \right] + 3 \left( \frac{1}{9} \right)^a + 8 \left( \frac{2}{11} \right)^a + 2 \left( \frac{1}{12} \right)^a \right] 2^{n+2}
\]

\[
- \left[ 4 \left( \frac{2}{9} \right)^a + 16 \left( \frac{1}{6} \right)^a + 24 \left( \frac{1}{7} \right)^a + 4 \left( \frac{2}{15} \right)^a + 8 \left( \frac{1}{9} \right)^a \right] + 24 \left( \frac{2}{11} \right)^a + 5 \left( \frac{1}{12} \right)^a. \quad (11)
\]
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Proof. Let \( G = TD_2[n] \). From equation (5) and Table 2, we derive

\[
HKV^a(TD_2[n]) = \sum_{uv \in E(G)} \left( \frac{2}{M_G(u) + M_G(v)} \right)^a
\]

\[
= \left( \frac{2}{2 + 3} \right)^a 2^{n+2} + \left( \frac{2}{3 + 6} \right)^a (2^{n+2} - 4) + \left( \frac{2}{3 + 8} \right)^a 2^{n+2}
\]

\[
+ \left( \frac{2}{6 + 6} \right)^a (7 \times 2^{n+2} - 16) + \left( \frac{2}{6 + 8} \right)^a (11 \times 2^{n+2} - 24)
\]

\[
+ \left( \frac{2}{6 + 9} \right)^a (2^{n+2} - 4) + \left( \frac{2}{6 + 12} \right)^a (3 \times 2^{n+2} - 8)
\]

\[
+ \left( \frac{2}{9 + 12} \right)^a (8 \times 2^{n+2} - 24) + \left( \frac{2}{12 + 12} \right)^a (2 \times 2^{n+2} - 5)
\]

\[
= \left[ \left( \frac{2}{5} \right)^a + \left( \frac{2}{9} \right)^a + \left( \frac{2}{11} \right)^a + 7 \left( \frac{1}{6} \right)^a + 11 \left( \frac{1}{7} \right)^a \right]
\]

\[
+ \left( \frac{2}{15} \right)^a + 3 \left( \frac{1}{9} \right)^a + 8 \left( \frac{2}{11} \right)^a + 2 \left( \frac{1}{12} \right)^a \right] 2^{n+2}
\]

\[
- \left[ 4 \left( \frac{2}{9} \right)^a + 16 \left( \frac{1}{6} \right)^a + 24 \left( \frac{1}{7} \right)^a + 4 \left( \frac{2}{15} \right)^a + 8 \left( \frac{1}{9} \right)^a \right]
\]

\[
+ 24 \left( \frac{2}{11} \right)^a + 5 \left( \frac{1}{12} \right)^a \right].
\]

Corollary 8.1. The harmonic KV index of \( TD_2[n] \) is

\[
HKV(TD_2[n]) = \left( \frac{11}{7} + \frac{5}{9} + \frac{18}{11} + \frac{28}{15} \right) 2^{n+2} + \left( \frac{25}{7} + \frac{40}{9} + \frac{48}{11} + \frac{5}{12} + \frac{8}{15} \right).
\]

Proof. Put \( a = 1 \) in equation (11), we obtain the desired result.

Theorem 9. The augmented KV index and its polynomial of \( TD_2[n] \) are given by

(i) \( AKVI(TD_2[n]) = \left[ \left( \frac{18}{7} \right)^3 + \left( \frac{8}{3} \right)^3 + 7 \left( \frac{18}{5} \right)^3 + \left( \frac{54}{13} \right)^3 + 8 \left( \frac{108}{19} \right)^3 \right] \)

\[
+ 2 \left( \frac{72}{11} \right)^3 + 2899 \right] 2^{n+2} - \left[ 4 \left( \frac{18}{7} \right)^3 + 16 \left( \frac{18}{5} \right)^3 + 4 \left( \frac{54}{13} \right)^3 \\
+ 24 \left( \frac{108}{19} \right)^3 + 5 \left( \frac{72}{11} \right)^3 + 7368 \right].
\]

(ii) \( AKVI(TD_2[n]) = 2^{n+2} x^8 + (2^{n+2} - 4)x \left( \frac{18}{7} \right)^3 + 2^{n+2} x \left( \frac{8}{3} \right)^3 \)
\[
+ (7 \times 2^{n+2} - 16)x \left( \frac{18}{5} \right)^3 + (11 \times 2^{n+2} - 24)x^64 \\
+ (2^{n+2} - 4)x \left( \frac{54}{13} \right)^3 + (3 \times 2^{n+2} - 8)x \left( \frac{9}{2} \right)^3 \\
+ (8 \times 2^{n+2} - 24)x \left( \frac{108}{19} \right)^3 + (2 \times 2^{n+2} - 5)x \left( \frac{72}{11} \right)^3.
\]

**Proof.** Let \( G \) be the graph of \( TD_2[n] \).

(i) By using equation (6) and Table 2, we obtain

\[
AKVI(TD_2[n]) = \sum_{uv \in E(G)} \left( \frac{M_G(u)M_G(v)}{M_G(u) + M_G(v) - 2} \right)^3
\]
\[
= \left( \frac{2 \times 3}{2 + 3 - 2} \right)^3 2^{n+2} + \left( \frac{3 \times 6}{3 + 6 - 2} \right)^3 (2^{n+2} - 4) + \left( \frac{3 \times 8}{3 + 8 - 2} \right)^3 2^{n+2}
\]
\[
+ \left( \frac{6 \times 6}{6 + 6 - 2} \right)^3 (7 \times 2^{n+2} - 16) + \left( \frac{6 \times 8}{6 + 8 - 2} \right)^3 (11 \times 2^{n+2} - 24)
\]
\[
+ \left( \frac{6 \times 9}{6 + 9 - 2} \right)^3 (2^{n+2} - 4) + \left( \frac{6 \times 12}{6 + 12 - 2} \right)^3 (3 \times 2^{n+2} - 8)
\]
\[
+ \left( \frac{9 \times 12}{9 + 12 - 2} \right)^3 (8 \times 2^{n+2} - 24) + \left( \frac{12 \times 12}{12 + 12 - 2} \right)^3 (2 \times 2^{n+2} - 5)
\]
\[
= \left[ \left( \frac{18}{7} \right)^3 + \left( \frac{8}{3} \right)^3 + 7 \left( \frac{18}{5} \right)^3 + \left( \frac{54}{13} \right)^3 + 8 \left( \frac{108}{19} \right)^3 + 2 \left( \frac{72}{11} \right)^3 + 2899 \right] 2^{n+2}
\]
(ii) From equation (7) and by using Table 2, we deduce

\[
AKVI(TD_2[n], x) = \sum_{uv \in E(G)} x^{M_G(u)M_G(v)}\left(\frac{M_G(u)M_G(v)}{M_G(u)+M_G(v)-2}\right)^3
\]

\[
= 2^{n+2} x^{\left(\frac{2\times3}{2+3-2}\right)^3} + (2^{n+2} - 4) x^{\left(\frac{3\times6}{3+6-2}\right)^3} + 2^{n+2} x^{\left(\frac{3\times8}{3+8-2}\right)^3}
\]

\[
+ (7 \times 2^{n+2} - 16) x^{\left(\frac{6\times6}{6+6-2}\right)^3} + (11 \times 2^{n+2} - 24) x^{\left(\frac{6\times8}{6+8-2}\right)^3}
\]

\[
+ (2^{n+2} - 4) x^{\left(\frac{6\times9}{6+9-2}\right)^3} + (3 \times 2^{n+2} - 8) x^{\left(\frac{6\times12}{6+12-2}\right)^3}
\]

\[
+ (8 \times 2^{n+2} - 24) x^{\left(\frac{9\times12}{9+12-2}\right)^3} + (2 \times 2^{n+2} - 5) x^{\left(\frac{12\times12}{12+12-2}\right)^3}.
\]

\[
= 2^{n+2} x^{8} + (2^{n+2} - 4) x^{\left(\frac{18}{7}\right)^3} + 2^{n+2} x^{\left(\frac{8}{3}\right)^3} + (7 \times 2^{n+2} - 16) x^\left(\frac{18}{5}\right)
\]

\[
+ (11 \times 2^{n+2} - 24) x^{64} + (2^{n+2} - 4) x^\left(\frac{54}{13}\right) + (3 \times 2^{n+2} - 8) x^\left(\frac{9}{2}\right)
\]

\[
+ (8 \times 2^{n+2} - 24) x^\left(\frac{108}{19}\right) + (2 \times 2^{n+2} - 5) x^\left(\frac{72}{11}\right).
\]

**Theorem 10.** The hyper F-KV index and its polynomial of \(TD_2[n]\) are given by

(i) \(HFKV(TD_2[n]) = 835588 \times 2^{n+2} - 2274720.\)

(ii) \(HFKV(TD_2[n], x) = 2^{n+2} x^{169} + (2^{n+2} - 4) x^{2025} + 2^{n+2} x^{5329}
\]

\[
+ (7 \times 2^{n+2} - 16) x^{5184} + (11 \times 2^{n+2} - 24) x^{10000}
\]

\[
+ (2^{n+2} - 4) x^{13689} + (3 \times 2^{n+2} - 8) x^{32400}
\]

\[
+ (8 \times 2^{n+2} - 24) x^{50625} + (2 \times 2^{n+2} - 5) x^{82944}.
\]
Proof. Let $G = TD_2[n]$.

(i) By using equation (8) and Table 2, we deduce

$$HFKV(TD_2[n]) = \sum_{uv \in E(G)} [M_G(u)^2 + M_G(v)^2]^2$$

$$= (2^2 + 3^2)^2 2^{n+2} + (3^2 + 6^2)^2 (2^{n+2} - 16) + (3^2 + 8^2)^2 2^{n+2}$$

$$+ (6^2 + 9^2)^2 (2^{n+2} - 4) + (6^2 + 12^2)^2 (3 \times 2^{n+2} - 8)$$

$$+ (9^2 + 12^2)^2 (8 \times 2^{n+2} - 24) + (12^2 + 12^2)^2 (2 \times 2^{n+2} - 5)$$

$$= 835588 \times 2^{n+2} - 2274720.$$ 

(ii) From equation (9), we have

$$HFKV(TD_2[n], x) = \sum_{uv \in E(G)} x^{[M_G(u)^2 + M_G(v)^2]^2}.$$ 

Then by using Table 2, we obtain

$$HFKV(TD_2[n], x) = 2^{n+2} x^{(3^2 + 3^2)^2} + (2^{n+2} - 4) x^{(3^2 + 6^2)^2}$$

$$+ 2^{n+2} x^{(3^2 + 8^2)^2} + (7 \times 2^{n+2} - 16) x^{(6^2 + 6^2)^2}$$

$$+ (11 \times 2^{n+2} - 24) x^{(6^2 + 8^2)^2} + (2^{n+2} - 4) x^{(6^2 + 9^2)^2}$$

$$+ (3 \times 2^{n+2} - 8) x^{(6^2 + 12^2)^2} + (8 \times 2^{n+2} - 24) x^{(9^2 + 12^2)^2}$$

$$+ (2 \times 2^{n+2} - 5) x^{(12^2 + 12^2)^2}$$

$$= 2^{n+2} x^{169} + (2^{n+2} - 4) x^{2025} + 2^{n+2} x^{5329}$$

$$+ (7 \times 2^{n+2} - 16) x^{5184} + (11 \times 2^{n+2} - 24) x^{10000}$$

$$+ (2^{n+2} - 4) x^{13689} + (3 \times 2^{n+2} - 8) x^{32400}$$

$$+ (8 \times 2^{n+2} - 24) x^{50625} + (2 \times 2^{n+2} - 5) x^{82944}.$$
References


