Variational Iteration Algorithm-I with an Auxiliary Parameter for Solving Fokker-Planck Equation

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Abstract

In this paper, variational iteration algorithm-I with an auxiliary parameter is implemented to investigate Fokker-Planck equations. To show the accuracy and reliability of the technique comparisons are made between the variational iteration algorithm-I with an auxiliary parameter and classic variational iteration algorithm-I. The comparison shows that variational iteration algorithm-I with an auxiliary parameter is more powerful and suitable method for solving Fokker-Planck equations. Furthermore, the proposed algorithm can successfully be applied to a large class of nonlinear and linear problems.

1. Introduction

The aim of this work is to apply the variational iteration algorithm-I [1] with an auxiliary parameter for the analytical treatment of the Fokker-Planck equation. The method is able to provide analytical results for nonlinear and linear problems, in a direct way very conveniently. One of the main characteristics of this method is that approximate solution of great accuracy can be obtained by only a few iterations. This method has a simple procedure, acceptable results and above all, this method can successfully be applied to a large class of linear and nonlinear problems [2]-[6].

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2. Variational Iteration Algorithm-I

Consider a general differential equation

\[ L[u(x)] + N[u(x)] = c(x), \]  

the terms \( L[u(x)] \) and \( N[u(x)] \) represent the linear and nonlinear term respectively, while \( c(x) \) is the inhomogeneous source term. Constructing a correction function for Eq. (1) as,

\[ u_{k+1}(x) = u_k(x) + \int_0^\lambda \lambda(\eta) \left[ L[u_k(\eta)] + N[u_k(\eta)] - c(\eta) \right] d\eta, \]  

where \( \lambda \) is a parameter, which is not known and called the Lagrange multiplier [7].

Taking the variation \( \delta \) on the one side as well as the other side of Eq. (2) with respect to \( u_k(x) \),

\[ \delta u_{k+1}(x) = \delta u_k(x) + \delta \int_0^\lambda \lambda(\eta) \left[ L[u_k(\eta)] + N[u_k(\eta)] - c(\eta) \right] d\eta, \]  

where \( u_k(\eta) \) is considered as a restricted term which means \( \delta u_k(\eta) = 0 \).

Using optimality conditions, the value of Lagrange multiplier \( \lambda(\eta) \) can be identified. An exact solution obtains when \( k \to \infty \).

\[ u(x) = \lim_{k \to \infty} u_k(x). \]  

In short, the formula for equation (1) is,

\[ \begin{align*} 
    u_0(x) & \text{ is an appropriate initial approximation,} \\
    u_{k+1}(x) & = u_k(x) + \int_0^\lambda \lambda(\eta) \left[ L[u_k(\eta)] + N[u_k(\eta)] - c(\eta) \right] d\eta \end{align*} \]  

This technique is called VIA-I, which is a further development of the general Lagrange multiplier technique for solving nonlinear problems by Inokuti et al. [7]. Now this method [8]-[10] has been developed [11-12] to solve a lot of problems arise in various fields of sciences.
3. Insertion of an Auxiliary Parameter in Variational Iteration Algorithm-I

In VIA-I, an auxiliary parameter $h$ can be inserted. The optimal choice of unknown $h$ improves the correctness, precision and effectiveness of the technique. After inserting $h$, equation (5) will become

$$
\begin{align*}
\left\{ u_0(x) \right\} & \text{ is an appropriate initial approximation}, \\
u_1(x, h) &= u_0(x) + h \int_0^x \lambda(\eta)[L[u_0(\eta)] + N[u_0(\eta)] - c(\eta)]d\eta, \\
u_{k+1}(x, h) &= u_k(x, h) + h \int_0^x \lambda(\eta)[L[u_k(\eta, h)] + N[u_k(\eta, h)] - c(\eta, h)]d\eta, \\
& \qquad k = 1, 2, 3, \ldots
\end{align*}
$$

This technique is known as VIA-I with AP. Actually, this technique is simple, has a lesser size of calculation, not difficult to analyze and have the ability to approximate the solution precisely in solution domain of wide range.

4. The Fokker-Planck Equation [2]

In this section, the general form of Fokker-Planck equation which is also called forward Kolmogorov equation is

$$
\frac{\partial u}{\partial t} = \left[ \frac{\partial}{\partial t} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u(x, t),
$$

with conditions:

$$
u(x, 0) = f(x), \quad x \in \mathcal{R}.
$$

It is the equation for the motion of concentration field $u(x, t)$. The backward Kolmogorov equation can be written in the following form

$$
\frac{\partial u}{\partial t} = -\left[ \frac{\partial}{\partial t} A(x, t) + \frac{\partial^2}{\partial x^2} B(x, t) \right] u(x, t).
$$

Let the initial conditions,

$$
u(x, 0) = f(x), \quad x \in \mathcal{R}
$$

and

$$
A(x, t) = -(x + 1),
$$
Then equation (8) becomes

\[
\frac{\partial u}{\partial t} = \left[ \frac{\partial^2}{\partial x^2} x^2 e^t - \frac{\partial}{\partial t} (x + 1) \right] u(x, t).
\] (9)

First, we solve this example by VIA-I.

Constructing the correction function for equation (9) as,

\[
u_{k+1}(x, t) = u_k(x, t) + \int_0^t \lambda(\eta) \left\{ \frac{\partial u_k(x, \eta)}{\partial \eta} + \frac{\partial u_k(x + \bar{t})}{\partial \eta} u_k(x, \eta) - \frac{\partial^2 (\bar{x}^2 e^{\eta})}{\partial x^2} u_k(x, \eta) \right\} d\eta.
\] (10)

Taking the variation \(\delta\) on the one side as well as the other side with respect to \(u_k(x, t)\)

\[
\delta u_{k+1}(x, t) = \delta u_k(x, t)
\]

\[
+ \delta \int_0^t \lambda(\eta) \left\{ \frac{\partial u_k(x, \eta)}{\partial \eta} + \frac{\partial u_k(x + \bar{t})}{\partial \eta} u_k(x, \eta) - \frac{\partial^2 (\bar{x}^2 e^{\eta})}{\partial x^2} u_k(x, \eta) \right\} d\eta.
\]

Ignoring the restricted terms

\[
\delta u_{k+1}(x, t) = \delta u_k(x, t) + \delta \int_0^t \lambda(\eta) \left\{ \frac{\partial u_k(x, \eta)}{\partial \eta} \right\} d\eta
\]

\[
= \delta u_k(x, t) + \lambda(\eta) \delta u_k(x, t) - \int_0^t \lambda'(\eta) \delta u_k(x, \eta) d\eta
\]

\[
= (1 + \lambda(\eta)) \delta u_k(x, t) - \int_0^t \lambda'(\eta) \delta u_k(x, \eta) d\eta.
\]

The stationary conditions are:

\[
\lambda'(\eta) = 0,
\]

\[
1 + \lambda(\eta) = 0,
\]

we get the value of \(\lambda(\eta)\) which is \(\lambda(\eta) = -1\).
Using this value of $\lambda(\eta)$ in equation (10) results in the below iterative scheme:

$$u_{k+1}(x, t) = u_k(x, t)$$

$$- \int_0^t \left( \frac{\partial u_k(x, \eta)}{\partial \eta} + \frac{\partial u_k(x+1, \eta)}{\partial \eta} u_k(x, \eta) - \frac{\partial^2 (x^2 e^\eta)}{\partial x^2} u_k(x, \eta) \right) d\eta. \quad (11)$$

starting with

$$u_0(x, t) = x + 1,$$

other approximations by using the scheme (11),

$$u_1(x, t) = (t + 1)(x + 1),$$

$$u_2(x, t) = \frac{(x + 1)(t^2 + 2t + 2)}{2},$$

$$u_3(x, t) = \frac{(x + 1)(t^3 + 3t^2 + 6t + 6)}{6},$$

we stop the procedure at $u_{10}(x, t)$. The absolute error of $u_{10}(x, t)$ in the solution domain $(x, t) \in [0, 5] \times [0, 1]$ can be seen in Figure 1.

**Figure 1.** Absolute error betwixt the exact and approximate solutions by VIA-I.
Now we want to solve this problem by VIA-I with AP.

Using VIA-I with AP, the recurrence relation for equation (9) is

\[
\begin{align*}
 u_{k+1}(x, t, h) &= u_k(x, t, h) - h \int_0^t \left\{ \frac{\partial u_k(x, \eta, h)}{\partial \eta} + \frac{\partial u_k(x + 1)}{\partial \eta} u_k(x, \eta, h) \\
&\quad - \frac{\partial^2 (x^2 \eta)}{\partial x^2} u_k(x, \eta, h) \right\} d\eta.
\end{align*}
\] (12)

Starting with

\[ u_0(x, t) = x + 1. \]

Other approximations can be get by using the recurrence relation (12),

\[
\begin{align*}
 u_1(x, t, h) &= (ht + 1)(x + 1), \\
 u_2(x, t, h) &= (ht + 1)(x + 1) + \frac{ht(x + 1)((ht - 2h + 2))}{2}, \\
 u_3(x, t, h) &= \frac{(x + 1)(h^3 t^3 - 6h^3 t^2 + 6h^3 t + 9h^2 t^2 - 18h^2 t + 18ht + 6)}{6}, \\
&\vdots
\end{align*}
\]

we stop the procedure at \( u_{10}(x, t, h) \).

The following residual function is defined

\[
\eta_{10}(x, t, h) = \frac{\partial u_{10}(x, t, h)}{\partial \eta} + \frac{\partial u_{10}(x + 1)}{\partial \eta} u_{10}(x, t, h) - \frac{\partial^2 (x^2 \eta)}{\partial x^2} u_{10}(x, t, h). \quad (13)
\]

The square of residual function for 10th-order approximation with respect to \( h \) for \((x, t) \in [0, 5] \times [0, 1]\) is

\[
\frac{1}{11^2} \sum_{i=0}^{10} \sum_{j=0}^{10} \left( \eta_{10} \left( \frac{5i}{10}, \frac{j}{10}, h \right) \right)^2. \quad (14)
\]

The minimum value of above square residual function occurs at \( h = 1.02408718538499 \). Using this value of \( h \) in \( u_{10}(x, t, h) \) in the solution domain \((x, t) \in [0, 5] \times [0, 1]\), error betwixt the exact and approximate solutions can be seen in Figure 2.
Comparing Figure 1 and Figure 2, it is clear that VIA-I with AP gives better results as compared to VIA-I. Numerical comparison between the exact and approximate solutions of both methods is given in the table below.

**Table 1.** Comparison of absolute errors for 6th order approximation by VIA-I and VIA-I with AP.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t$</th>
<th>Absolute Error in VIA-I with AP</th>
<th>Absolute Error in VIA-I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>3.553×10^{-16}</td>
<td>2.220×10^{-16}</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2</td>
<td>2.665×10^{-15}</td>
<td>1.332×10^{-15}</td>
</tr>
<tr>
<td>1.5</td>
<td>0.3</td>
<td>2.398×10^{-14}</td>
<td>1.137×10^{-13}</td>
</tr>
<tr>
<td>2.0</td>
<td>0.4</td>
<td>7.994×10^{-15}</td>
<td>3.261×10^{-12}</td>
</tr>
<tr>
<td>2.5</td>
<td>0.5</td>
<td>6.750×10^{-14}</td>
<td>4.467×10^{-11}</td>
</tr>
<tr>
<td>3.0</td>
<td>0.6</td>
<td>4.174×10^{-13}</td>
<td>3.826×10^{-10}</td>
</tr>
<tr>
<td>3.5</td>
<td>0.7</td>
<td>3.608×10^{-12}</td>
<td>2.376×10^{-09}</td>
</tr>
<tr>
<td>4.0</td>
<td>0.8</td>
<td>3.947×10^{-11}</td>
<td>1.152×10^{-08}</td>
</tr>
<tr>
<td>4.5</td>
<td>0.9</td>
<td>6.701×10^{-10}</td>
<td>4.672×10^{-08}</td>
</tr>
<tr>
<td>5.0</td>
<td>1.0</td>
<td>4.997×10^{-09}</td>
<td>1.639×10^{-07}</td>
</tr>
</tbody>
</table>
The above table shows that VIA-I with AP is better for a large domain of $t$ as compared to VIA-I.

5. Conclusions

In this paper, variational iteration algorithm-I with an auxiliary parameter has been used in a way that accomplished the desired aim for solving Fokker-Planck equation. This work has made sure that the variational iteration algorithm-I with an auxiliary parameter offers noteworthy advantages in terms of its easy applicability, its computational success, and its adequacy to solve a wide class of differential equations. Graphical and numerical results reveal that this modification of variational iteration algorithm-I is suitable for all linear and nonlinear problems arise in physical sciences and engineering, superior to the variational iteration algorithm-I.

References


